INVARIANTS FOR 1-DIMENSIONAL COHOMOLOGY CLASSES ARISING FROM TQFT

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Abstract. Let (V, Z) be a Topological Quantum Field Theory over a field f defined on a cobordism category whose morphisms are oriented $n+1$-manifolds perhaps with extra structure (for example a p_1 structure and banded link). Let (M, χ) be a closed oriented $n+1$-manifold M with this extra structure together with $\chi \in H^1(M)$. Let M_∞ denote the infinite cyclic cover of M given by χ. Consider a fundamental domain E for the action of the integers on M_∞ bounded by lifts of a surface Σ dual to χ, and in general position. E can be viewed as a cobordism from Σ to itself. We give Turaev and Viro's proof of their theorem that the similarity class of the non-nilpotent part of $Z(E)$ is an invariant. We give a method to calculate this invariant for the (V_p, Z_p) theories of Blanchet, Habegger, Masbaum and Vogel when M is zero framed surgery to S^3 along a knot K. We give a formula for this invariant when K is a twisted double of another knot. We obtain formulas for the quantum invariants of branched covers of knots, and unbranched covers of 0-surgery to S^3 along knots. We study periodicity among the quantum invariants of Brieskorn manifolds. We give an upper bound on the quantum invariants of branched covers of fibered knots. We also define finer invariants for pairs (M, χ) for TQFT's over Dedekind domains. We use these ideas to study isotopy invariants of banded links in $S^1 \times S^2$.

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Introduction

Witten conceived of topological quantum field theory and related it to the Jones polynomial [W]. Axioms, based on Segal's axioms for conformal field theory, were given by Atiyah [A1,A2]. The first rigorous development of the projective version of the TQFT's which most concern us here was given by Reshetikhin-Turaev [RT]. There are now several rigorous mathematical approaches to topological quantum field theory. We have used the work of Blanchet, Habegger, Masbaum and Vogel [BHMV1,MV1] as a foundation for our work, as it is the most complete development of the subject for someone with our background. We have also been influenced by the papers of Lickorish [L1] and Walker [Wa1] which will not be explicitly referred to below. If we have a $n+1$-manifold fibered over a circle, and a TQFT in $n+1$ dimensions, then the monodromy induces an automorphism of the vector space associated to the fiber. This construction was generalized to $n+1$-manifolds M together with a one dimensional cohomology class χ by Turaev and Viro [TV] as follows. One considers a fundamental domain E for the action of the integers on M_∞, the infinite cyclic cover of M. E can be viewed as a cobordism from a surface

to itself. We give Turaev and Viro's proof of their theorem that the similarity class of the non-nilpotent part of the induced endomorphism is an invariant.

In §1, we describe the Turaev-Viro module of (M, χ), which Turaev and Viro conceived of as being somewhat analogous to the the Alexander module of a knot, but with a TQFT replacing homology. We study a number of properties of the Turaev-Viro module and its associated invariants. In particular, we relate these invariants to TQFT invariants of the finite cyclic covers of M given by χ. In §2, we show how this invariant may be refined if we are working with a TQFT defined over a Dedekind domain, rather than a field. The results of the first two sections are axiomatic and apply to any TQFT and more generally to many linearizations of cobordism categories. By a linearization over a ring d, we mean a functor from a cobordism category to a category of modules over d. If target category is a category of finitely generated modules over d, the linearization is said to be finite. In particular, §1 and §2 may be applied to the Z_p theories of [BHMV1] and the theories of Frohman and Nicas [FN1,FN2]). In §3, we discuss various issues involving V_p theories and p_1-structures. In §4 we study banded links L in $S^1 \times S^2$ which are null homologous modulo two. We define a restricted cobordism category $\mathcal{C}$ and a finite linearization of $\mathcal{C}$ over $\mathbb{Z}[A, A^{-1}]$. We obtain in this way a polynomial invariant $D(L) \in \mathbb{Z}[A, A^{-1}]$, as well as an invariant of L which is a similarity class of automorphisms of modules over $\mathbb{Q}[A, A^{-1}, \frac{1}{D(L)}]$. For almost all p, these invariants specialize to invariants associated to L by the Z_p theory as above. In §5, we adapt Rolfsen's method [R] of calculating the Alexander module to the problem of calculating the Turaev-Viro module associated to 0-framed surgery along a knot. We study twisted doubles of knots in detail. We also calculate the invariant for the knot 8_8 for $p = 5$. We also prove a number of general results. We use a result of Casson and Gordon to obtain a restriction on this invariant for a fibered ribbon knot.

In §6, we use these results to calculate the quantum invariant $< >_p$ for the finite cyclic covers of 0-framed surgery along knots. In §7, we introduce certain colored invariants of knots which are necessary to give a good formula for the Turaev-Viro modules of a connected sum. These same colored invariants are then used in §8 to give formulas for the quantum invariants of the branched cyclic covers of knots. We use these formulas to give closed formulas for $< >_5$ for all the branched cyclic covers of the trefoil, the figure eight, the stevedore's knot, and the untwisted double of the figure eight. In fact, using our formulas, it is an easy matter to calculate recursively $< >_5$ for all the branched cyclic covers of a twisted double of a knot J, once we know two values of the Kauffman polynomial of J. The same may be said for the unbranched cyclic covers of zero surgery to S^3 along a twisted double of J. We then derive some periodicity results for quantum invariants of Brieskorn manifolds and more generally branched covers of fibered knots with periodic monodromy. For fibered knots whose monodromy is not necessarily periodic, we obtain an upper bound for their quantum invariants. In §9, we consider the extent to which these invariants are skein invariants. In §10, we discuss $V_p(\Sigma)$ for p odd and Σ disconnected. In §11, we compare when possible our calculations with other calculations and methods. The afterword has some final conjectures and other remarks. In the interest of the reader, we will frequently derive a result, and then subsequently derive a more general result by a more difficult proof, or a proof requiring more background. We used Mathematica running on a NeXT computer for

our calculations.

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§1 The Turaev-Viro module

Suppose we have a concrete cobordism category C in the sense of [BHMV1,(1.A)]. The objects of C are compact oriented manifolds of dimension n with perhaps some extra structure. A morphism from Σ to Σ' is a compact oriented manifold M of dimension $n+1$ with perhaps some extra structure together with a diffeomorphism of ∂M to the disjoint union $-\Sigma \coprod \Sigma'$, up to equivalence. We call such a manifold a cobordism from Σ to Σ'. Two such cobordisms are equivalent if there is a diffeomorphism between them respecting the diffeomorphism of the boundary with $-\Sigma \coprod \Sigma'$. In other words, the relevant diagram must commute. Moreover the extra structure on M must induce the extra structure on $-\Sigma \coprod \Sigma'$. We assume codimension-zero submanifolds in general position inherit this structure. In addition we assume the compact codimension zero submanifolds in general position in an infinite cyclic covering space of a $n+1$-manifold with this structure inherit such a structure from their base. An especially important example is the category $C_2^{p_1}$ whose objects are closed smooth 2-manifolds with p_1 structure containing a banded (an interval passing through each point) collection of points and whose morphisms are smooth 3-manifolds with p_1 structure containing a banded link [BHMV1]. A banded link in a 3-manifold is an embedded oriented surface diffeomorphic to the product of a 1-manifold with an interval which meets the boundary of the 3-manifold in the product of the boundary of the 1-manifold with the interval.

Let f be a field with involution $\lambda \to \bar{\lambda}$ (perhaps trivial). Next suppose we have finite linearization of C over f. It assigns a k-vector space $V(\Sigma)$ to an object Σ and a linear transformation $Z(M)$ to a morphism M. We let (V, Z) denote this functor. If we were to follow the notation of [BHMV1,(1.A)], we should denote $Z(M)$ by Z_M, and reserve $Z(M)$ for the morphism induced by the manifold M viewed as a cobordism from $\emptyset$ to ∂M. However this gets more difficult to read as subscripts proliferate, and it will be clear from context how we are thinking of M as a cobordism from which part of the boundary to which other part of the boundary.

Let M_∞ denote the infinite cyclic cover of M classified by χ, let π denote the projection and T denote the generating covering transformation. Suppose γ is a path covering a loop on which χ evaluates to 1, then our convention is that $\gamma(1) = T(\gamma(0))$. Let $\tilde{\Sigma}$ be any lift of Σ in M_∞. Let $E(\tilde{\Sigma})$ be the compact submanifold of M_∞ with boundary $-\tilde{\Sigma} \coprod T\tilde{\Sigma}$ which may view as a cobordism from $\tilde{\Sigma}$ to $T\tilde{\Sigma}$. $E(\tilde{\Sigma})$ is a fundamental domain for the action of the integers on M_∞. If we take Σ to be in general position, $E(\tilde{\Sigma})$ defines a morphism in C. Since the projection π defines a specific diffeomorphism from any lift of Σ and Σ preserving any extra structure, we may regard $Z(E(\tilde{\Sigma}))$ as an endomorphism of $V(\Sigma)$. Note that $E(\tilde{\Sigma})$ is diffeomorphic to the exterior of Σ, i.e. M minus an open tubular neighborhood of Σ. However this diffeomorphism does not necessarily preserve extra structure.

Alternatively we may define $E(\Sigma)$ to be M "slit" along Σ. This is the $n+1$-manifold obtained by replacing a tubular neighborhood of Σ by $\Sigma \times [-1, 0] \coprod \Sigma \times [0, 1]$. In other words, we take M after we have replaced each point of Σ by two points and defined neighborhood systems for these points appropriately. $E(\Sigma)$ is a

$n+1$-manifold with structure with boundary $-\Sigma \coprod \Sigma$.

Given a linear endomorphism $\mathcal{Z}$ of a finite dimensional vector space $\mathcal{V}$, $\mathcal{V}$ has a canonical $\mathcal{Z}$-invariant direct sum decomposition as $\mathcal{V}_0 \oplus \mathcal{V}_\flat$, where $\mathcal{Z}$ restricted to $\mathcal{V}_0$ is nilpotent and $\mathcal{Z}$ restricted to $\mathcal{V}_\flat$ is an automorphism, denoted $\mathcal{Z}_\flat$. Here $\mathcal{V}_0 = \cup_{k\geq 1}\text{Kernel}(\mathcal{Z}^k)$, and $\mathcal{V}_\flat = \cap_{k\geq 1}\text{Image}(\mathcal{Z}^k)$. We let $\mathcal{M}(\mathcal{Z})$ denote $\mathcal{V}_\flat$ viewed as a $f[t,t^{-1}]$-module where t acts by $\mathcal{Z}_\flat$. We observe that one actually has $\mathcal{V}_0 = \text{Kernel}(\mathcal{Z}^{\dim(\mathcal{V})})$, and $\mathcal{V}_\flat = \text{Image}(\mathcal{Z}^{\dim(\mathcal{V})})$. Note $\dim(\mathcal{V}_\flat)$ is simply the number of nonzero eigenvalues of $\mathcal{Z}$ counted with multiplicity. $\mathcal{V}_0$ is also known as the generalized 0-eigenspace.

Let $\mathcal{M}_Z(M,\chi)$ denote the $k[t,t^{-1}]$-module $\mathcal{M}(Z(E(\Sigma)))$. Let $\mathcal{A}_Z(M,\chi)$ denote the automorphism $Z(E(\Sigma))_\flat$. In most cases, we will drop the subscript Z.

Theorem (1.1)(Turaev and Viro). *The module $\mathcal{M}(M,\chi)$ is a well defined invariant of the pair (M,χ) up to isomorphism. In other words, $\mathcal{A}(M,\chi)$ is well defined up to similarity class.*

Proof. Let Σ, and $\tilde\Sigma$ be as above, and let $E = E(\tilde\Sigma)$. For k an integer, define $\tilde\Sigma_k = T^k\tilde\Sigma$. For a negative integer $k<0$, let $E_k = \cup_{k\leq i\leq -1}T^iE$. For a positive integer k, let $E_k = \cup_{0\leq i\leq k-1}T^iE$. Of course $\tilde\Sigma_k$ has a natural diffeomorphism with Σ. We can use this diffeomorphism to give a decomposition $V(\tilde\Sigma_k) = V(\tilde\Sigma_k)_0 \oplus V(\tilde\Sigma_k)_\flat$. If Σ' is a second oriented surface dual to χ in general position and, let $\tilde\Sigma'$ be any lift of Σ' which is disjoint from $\tilde\Sigma$ and which lies in $\cup_{i\geq 0}T^iE$. We consider also the case that $\Sigma' = \Sigma$, but $\tilde\Sigma' \neq \tilde\Sigma$. Let W be the compact submanifold of M_∞ with boundary $-\tilde\Sigma \coprod \tilde\Sigma'$. Define E'_k, $\tilde\Sigma'_k$ and $V(\tilde\Sigma'_k)_\flat$ analogously to the unprimed items. The result will follow from three lemmas:

Lemma (1.2). *$Z(W): V(\tilde\Sigma) \to V(\tilde\Sigma')$ will send $V(\tilde\Sigma)_\flat$ to $V(\tilde\Sigma')_\flat$.*

Proof. Suppose that $x \in V(\tilde\Sigma)_\flat$, and $Z(W)x = x'$. For every $k<0$, there is a $y \in V(\tilde\Sigma_k)$ such that $Z(E_k)y = x$. Let $Z(T^{-k}W)y = y'$. Since $T^{-k}W \cup E'_k = E_k \cup W$, we have that $Z(E'_k)y' = y$. Thus we have proved that $Z(W): V(\tilde\Sigma) \to V(\tilde\Sigma')$ will send $V(\tilde\Sigma)_\flat$ to $V(\tilde\Sigma')_\flat$. We will denote this homomorphism $Z(W)_\flat$. □

Lemma (1.3). *$Z(W)_\flat$ sends $V(\tilde\Sigma)_\flat$ isomorphically to $V(\tilde\Sigma')_\flat$*

Proof. For some negative k, $\tilde\Sigma'_k$ will lie in $\cup_{i<0}T^i(E(\Sigma))$. Let X be the compact submanifold of M_∞ with boundary $-\tilde\Sigma'_k \coprod \tilde\Sigma$, then $X \cup W = E'_k$. By functoriality, $Z(W)_\flat \circ Z(X)_\flat = Z(E'_k)_\flat$. $Z(E'_k)$ may be naturally identified with $Z(E')^k$, which is an isomorphism when restricted to $V(\Sigma')_\flat$. Thus $Z(E'_k)_\flat$ is an isomorphism. Thus $Z(W)_\flat$ is surjective.

Similarly $W \cup T^{-k}X = E_{-k}$, and $Z(E_{-k})_\flat$ is an isomorphism. By functoriality, $Z(T^{-k}X)_\flat \circ Z(W)_\flat = Z(E_k)_\flat$, thus $Z(W)_\flat$ is injective. □

Lemma (1.4). *Assume $\tilde\Sigma'$ is a lift of Σ' which lies in $\cup_{i\geq 1}T^i(E(\Sigma))$. Let U be the compact submanifold of M_∞ with boundary $-T\tilde\Sigma \coprod \tilde\Sigma'$. $Z(U)_\flat \circ Z(\tilde E)_\flat = Z(E'_{-1})_\flat \circ Z(T^{-1}U)_\flat$. Thus $Z(E)_\flat$ is similar to $Z(E')_\flat$.*

Proof. $\tilde E \cup U = t^{-1}U \cup E'_{-1}$. □

Remarks. Walker earlier observed that $\text{rank}(Z(\tilde E)_\flat)$ $(= \dim_f(\mathcal{M}(M,\chi)))$ is an invariant [Wa2]. Suppose $\dim V(\Sigma)$ for Σ connected depends only on the genus of Σ. Suppose that $\dim V(\Sigma)$ is a increasing function of the genus Σ, as is true for

Witten's TQFT's. Then $\dim_f(\mathcal{M}(M,\chi))$ can be used to give lower bounds on the least genus of an embedded surface dual to χ. This is just the Thurston norm on $H_2(M)$ [T]. Walker discussed this application. Turaev and Viro then strengthened Walker's work.

Let $\Gamma_Z(M,\chi)$ denote the characteristic polynomial of $\mathcal{A}(M,\chi)$. We define the normalized characteristic polynomial of a matrix or endomorphism of a free module to be the characteristic polynomial in x divided by the highest power of x dividing this polynomial. $\Gamma_Z(M,\chi)$ is then the normalized characteristic polynomial of $Z(E)$ for any choice of Σ dual to χ in M. It will be convenient to let $D(M,\chi)$ denote the constant term of $\Gamma_Z(M,\chi)$.

Our convention is that the characteristic polynomial of an endomorphism of a zero dimensional vector space is the constant 1. We have $\deg(\Gamma_Z(M,\chi)) = \dim_f(\mathcal{M}((M,\chi)))$. $\Gamma_Z(M,\chi)$ is analogous to the Alexander polynomial, and should be called the Turaev-Viro polynomial. A complete set of invariants for $\mathcal{M}(M,\chi)$ is of course given by the invariant factors of $\mathcal{A}(M,\chi)$. These are in turn determined by certain determinantal divisors [AW,p.312] analogous to the higher Alexander polynomials. Two matrices are similar over f if and only if they are similar over a larger field [AW,p.315]. Thus no information is lost if we extend our scalars to a larger field. Thus another complete invariant would be the Jordan form of $\mathcal{A}(M,\chi)$ over the algebraic closure of f.

For the rest of this section, we suppose that (V,Z) *is a cobordism generated quantization i.e.* (V,Z) *satisfies axioms Q1 Q2 and CG of* [BHMV1]. Given a vector space $\mathcal{V}$ over f, then $\mathcal{V}^*$ denotes the vector space with the same underlying Abelian group but with $\lambda v \in \mathcal{V}^*$ given by $\bar{\lambda} v \in \mathcal{V}$ for $\lambda \in f$ and $v \in \mathcal{V}$. If $\mathcal{M}$ is a module over $f[t,t^{-1}]$, then $\mathcal{M}^*$ denotes the conjugate of the underlying vector space, with t acting the same as before on the underlying Abelian group. It is clear that a matrix for the action of t on $\mathcal{M}^*$, is given by the conjugate of a matrix for the action of t on $\mathcal{M}$. Making use of the fact that a matrix is similar to its transpose, we have the following proposition:

Proposition (1.5). *We have:*

$$\mathcal{M}(M,-\chi) = \mathcal{M}(M,\chi)$$

$$\mathcal{M}(-M,\chi) = \mathcal{M}(M,\chi)^*.$$

Proposition (1.6). *Suppose* M *is a fiber bundle over circle with fiber* Σ *and let* $\chi \in H^1(M)$ *be the cohomology class which is classified by the projection, then* $D(M,\chi)\overline{D(M,\chi)} = 1$. *If* z *is an eigenvector of* $\mathcal{A}(M,\chi)$ *with eigenvalue* λ, *either* $< z,z >_\Sigma = 0$, *or* $\lambda\bar{\lambda} = 1$. *Moreover eigenvectors with distinct eigenvalues are orthogonal with respect* $< \ , \ >_\Sigma$. *In particular if the inner product is definite, then all the roots of* $\Gamma(M,\chi)$ *have their conjugates as reciprocals.*

Proof. The monodromy T of this bundle is a diffeomorphism of Σ which preserves the p_1-structure which Σ inherits from M. $E(\Sigma) \in \tilde{\mathcal{M}}$ is the mapping cylinder of T. One may check that T is an isometry of $< \ , \ >_\Sigma$. Thus the norm of the determinant of a matrix which represents this map is one. Then $< z,z >_\Sigma = < Tz,Tz >_\Sigma = \lambda\bar{\lambda} < z,z >_\Sigma$. Similarly if z_1 and z_2 are eigenvectors with distinct eigenvalues λ_1 and λ_2, $< z_1,z_2 >_\Sigma = < Tz_1,Tz_2 >_\Sigma = \lambda_1\bar{\lambda}_2 < z_1,z_2 >_\Sigma$. □

For the rest of this section, we suppose that (Z, V) *satisfies all the axioms for a TQFT in the sense of* [BHMV1,(1.A)]. Let χ_d denote χ modulo d and $(M,\chi)_d$ denote the d-fold cyclic cover of M classified by χ_d with the induced structure. By the trace formula of TQFTs [BHVM, (1.2)], we have:

Proposition (1.7). *We have:* $Z((M,\chi)_d) = \mathit{Trace}(\mathcal{A}((M,\chi))^d)$.

Corollary (1.8). $Z((M,\chi)_d)$ *may be computed recursively with recursion relation given by* $\Gamma(M,\chi)$. *If* f *has characteristic zero, then the values of* $Z((M,\chi)_d)$ *for all* d, *determine* $\Gamma(M,\chi)$.

Proof. The first statement just follows from the Cayley-Hamilton Theorem applied to $\mathcal{A}(M,\chi)$, or it may be obtained from Newton's formula as below. The trace of $\mathcal{A}(M,\chi))^d$ is the sum of the dth powers of the eigenvalues of $\mathcal{A}(M,\chi)$ counted with multiplicity. Moreover the coefficients of $\Gamma(M,\chi)$ are, up to sign, the elementary symmetric functions of the eigenvalues of $\mathcal{A}(M,\chi)$ counted with multiplicity. Thus the initial terms of the sequence $Z((M,\chi)_d)$ may also be computed from the coefficients of $\Gamma(M,\chi)$ using Newton's formula [Ms, (Problem 16-A)]. Over a field of characteristic zero, Newton's formula allows us to calculate the coefficients of $\Gamma(M,\chi)$ recursively from the values of $Z((M,\chi)_d)$. $\square$

Remarks. For example: if $\Gamma(M,\chi) = x^2 - \sigma_1 x + \sigma_2$ then $Z(M) = \sigma_1$, $Z((M,\chi)_2) = \sigma_1^2 - 2\sigma_2$, and $Z((M,\chi)_d) = \sigma_1 Z((M,\chi)_{d-1}) - \sigma_2 Z((M,\chi)_{d-2}$ for $d > 2$. Girard's Formula [MS, (Problem 16-A)] gives a closed formula for $Z((M,\chi)_d)$ in terms of the coefficients of $\Gamma(M,\chi)$. Thus $\Gamma(M,\chi)$ contains the same information as the sequence $Z((M,\chi)_d)$. However $\Gamma(M,\chi)$ is a compact way of organizing this information. If $\Gamma(M,\chi)$ has distinct roots, then it determines the similarity class of $\mathcal{A}(M,\chi)$ and thus the isomorphism class of $(\mathcal{M},\chi)$.

Proposition (1.9). *Suppose* (V, Z) *is a TQFT defined on* $C_2^{p_1}$ *and it satisfies the surgery axiom (S1) of* [BHMV1], *then we have:*

$$\mathcal{A}(M\#M', \chi \oplus \chi') = \eta(\mathcal{A}(M,\chi) \otimes \mathcal{A}(M',\chi')).$$

§2 TQFT's over Dedekind Domains

For each positive integer p, [BHMV1] defined cobordism generated quantizations (V_p, Z_p) which take values in free finitely generated k_p modules and homomorphism of k_p modules, where

$$k_p = \mathbb{Z}[\frac{1}{d}, A, \kappa]/(\varphi_{2p}(A), \kappa^6 - u)$$

where $\varphi_{2p}(A)$ is the $2p$-cyclotomic polynomial in the indeterminate A,

$$d = \begin{cases} p, & \text{for } p \neq 3,4,6 \\ 1, & \text{for } p = 3,4 \\ 2, & \text{for } p = 6, \end{cases} \quad \text{and } u = \begin{cases} A^{-6-\frac{p(p+1)}{2}}, & \text{for } p \neq 1,2 \\ 1, & \text{for } p = 1 \\ A, & \text{for } p = 2. \end{cases}$$

We let A_p denote the image of A in k_p. Note that u_p, the image of u in k_p, is also 1 for $p = 3$ or 4.

In every case k_p is the ring of integers of a cyclotomic number field localized with respect to the multiplicative subset $\{d^n \,|n \in \mathbb{Z},\ n \geq 0\}$ and so is a Dedekind domain by [La,p.21] for instance. For this reason, we consider now invariants which may be defined in this, and in even more general circumstances which are potentially stronger than those obtained by passing to the field of fractions and applying the previous section. *In this section we will assume that* (V, Z) *is a finite linearization over a Dedekind domain* k.

Let $\mathcal{V}$ be a finitely generated module over a Dedekind domain k, $\mathcal{Z}$ a endomorphism of $\mathcal{V}$. Let $\mathcal{V}_0 = \cup_{k\geq 1}\mathrm{Kernel}(\mathcal{Z}^k)$. Let $\mathcal{V}_\sharp = \mathcal{V}/\mathcal{V}_0$. Since $\mathcal{V}_0$ is $\mathcal{Z}$ invariant, there is an induced endomorphism $\mathcal{Z}_\sharp$ of $\mathcal{V}_\sharp$. It is clear that $\mathcal{Z}_\sharp$ is injective. Let $coker(\mathcal{Z}_\sharp)$ denote the cokernel of $\mathcal{Z}_\sharp$. Because $\mathcal{Z}_\sharp$ is injective, $coker(\mathcal{Z}_\sharp)$ is a torsion module. To see this consider the map induced on $\mathcal{V}$ modulo its torsion submodule. This map is an isomorphism after tensoring with the field of fractions of k. It follows that $\mathrm{coker}(\mathcal{Z}_\sharp)$ is a torsion module. It must also be finitely generated as $\mathcal{V}$ maps onto it. Thus $\mathrm{coker}(\mathcal{Z}_\sharp)$ is a direct sum of cyclic modules of the form k/ann where ann is a non-trivial ideal of k [J,Thm. 10.15].

A k-module is of finite length if and only if it is finitely generated. This follows from the classification of finitely generated modules over a Dedekind domain given in [J, Chapt.10]. Given k-module F of finite length, Serre [S,p.14] defined an ideal of k, denoted $\chi_k(F)$. Given a short exact sequence, the value of χ_k on the middle term is the product of its value on the side terms. $\chi_k(G)$, for G a torsion module, is a nontrivial ideal. In fact if

$$G = \bigoplus_{\mathfrak{p},i} \mathfrak{p}^{e_{i,\mathfrak{p}}}$$

then

$$\chi_k(G) = \prod_{\mathfrak{p}} \mathfrak{p}^{(\sum_i e_{i,\mathfrak{p}})},$$

where $\mathfrak{p}$ ranges over all prime ideals of k, and almost all $e_{i,\mathfrak{p}}$ are zero. If k is a PID, then $\chi_k(F)$ is the order ideal of F as defined by Milnor [M]. If G is the cokernel of a 1-1 map of free k modules of finite rank, $\chi_k(G)$ has a nice interpretation. We need the following result which is less general than [Ba,p. 500]. We include a proof for the convenience of the reader.

Proposition (2.1). *If* G *is the cokernel of* $\mathcal{Z} : k^n \to k^n$ *and* $\det(\mathcal{Z}) \neq 0$, *then* $\chi_k(G)$ *is the principal ideal generated by* $\det(\mathcal{Z})$.

Proof. We need to see that for each $\mathfrak{p}$, $\nu_{\mathfrak{p}}(\det u) = \sum_i e_{i,\mathfrak{p}}$. Clearly $\det(u)_{\mathfrak{p}} = \det(u_{\mathfrak{p}})$. Also $\mathrm{cokernel}(u_{\mathfrak{p}}) = \oplus_i \mathfrak{p}^{e_{i,\mathfrak{p}}}$. Thus $\chi_{k_{\mathfrak{p}}}(\mathrm{cokernel}(u_{\mathfrak{p}})) = \mathfrak{p}^{\sum_i e_{i,\mathfrak{p}}}$. Finally as $k_{\mathfrak{p}}$ is a PID and the result to be proved is true for PID's [S,p.17 Lemma 3], $\det(u_{\mathfrak{p}}) = \chi_{k_{\mathfrak{p}}}(\mathrm{cokernel}(u_{\mathfrak{p}}))$. $\square$

Let $\mathcal{I}(\mathcal{Z})$ denote $\chi_k(\mathrm{coker}(\mathcal{Z}_\sharp))$. Let f denote the field of fraction of k, and let $k_{\mathcal{I}}$ denote ring $\{x \in f \mid \nu_{\mathfrak{p}}(x) \geq 0\ \forall \mathfrak{p} \nmid \mathcal{I}\}$. Let $\mathcal{Z}_\natural$ denote the endomorphism $\mathcal{Z}_\sharp \otimes \mathrm{id}_{k_{\mathcal{I}(\mathcal{Z})}}$, of $\mathcal{V}_\sharp \otimes k_{\mathcal{I}(\mathcal{Z})}$, which we denote $\mathcal{V}_\natural$. Then $\mathcal{Z}_\natural$ is an isomorphism. Here we have localized as little as possible such that $\mathcal{Z}_\natural$ is an isomorphism. Let $\mathfrak{M}(\mathcal{Z})$ denote the $k_{\mathcal{I}(\mathcal{Z})}[t, t^{-1}]$-module given by the action of $\mathcal{Z}_\natural$ on $\mathcal{V}_\natural$.

An $n \times n$ matrix H with coefficients in k defines an endomorphism $\mathcal{Z}_H$ of k^n. We let $H_\natural$ denote the similarity class of the induced automorphism $(\mathcal{Z}_H)_\natural$ of a $k_{\mathcal{I}(\mathcal{Z}_H)}$-module.

Proposition (2.2). *Let $\mathcal{Z}$ and $\mathcal{Z}'$ be two endomorphisms defined on the finitely generated k-modules $\mathcal{V}$ and $\mathcal{V}'$. Suppose that the induced injections $\mathcal{Z}_\sharp$, and $\mathcal{Z}'_\sharp$ fit into commutative diagram with α injective:*

$$\begin{array}{ccc} \mathcal{V}_\sharp & \xrightarrow{\mathcal{Z}_\sharp} & \mathcal{V}_\sharp \\ {\scriptstyle\alpha}\downarrow & & \downarrow{\scriptstyle\alpha} \\ \mathcal{V}'_\sharp & \xrightarrow{\mathcal{Z}'_\sharp} & \mathcal{V}'_\sharp \end{array}$$

Then $\chi_k(coker(Z_\sharp)) = \chi_k(coker(Z'_\sharp))$.

Proof. We form a short exact sequence of chain complexes all of whose nonzero terms are concentrated in two dimensions:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathcal{V}_\sharp & \xrightarrow{\mathcal{Z}_\sharp} & \mathcal{V}_\sharp & \longrightarrow & \text{coker}(\mathcal{Z}_\sharp) & \longrightarrow & 0 \\ & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & \\ 0 & \longrightarrow & \mathcal{V}'_\sharp & \xrightarrow{\mathcal{Z}_\sharp} & \mathcal{V}'_\sharp & \longrightarrow & \text{coker}(\mathcal{Z}'_\sharp) & \longrightarrow & 0 \end{array}$$

Then the induced long exact sequence of homology is:

$$0 \to \ker(\beta) \to \text{coker}(\alpha) \to \text{coker}(\alpha) \to \text{coker}(\beta) \to 0.$$

On the other hand we have:

$$0 \to \ker(\beta) \to \text{coker}(\mathcal{Z}_\sharp) \to \text{coker}(\mathcal{Z}'_\sharp) \to \text{coker}(\beta) \to 0.$$

Because of the multiplicative property of χ_k, the result follows. $\square$

Let $\mathfrak{I}_Z(M,\chi)$ denote the ideal $\mathcal{I}(Z(E)$. Let $\mathfrak{M}_Z(M,\chi)$ denote the module $\mathfrak{M}(Z(E(\Sigma)))$. Let $\mathfrak{A}_Z(M,\chi)$ denote the automorphism $Z(E(\Sigma))_\natural$. In most cases, we will drop the subscript Z.

Theorem (2.3). *$\mathfrak{I}(M,\chi)$, and the isomorphism class of $\mathfrak{M}(M,\chi)$ (or the similarity class of $\mathfrak{A}(M,\chi)$) are invariants of (M,χ).*

Proof. We use the notations at the beginning of the proof of Theorem (1.1). Let $V(\Sigma_k)_\sharp = V(\Sigma_k)/V(\Sigma_k)_0$. Analogous to Lemma(1.2) we have:

Lemma(2.4). *$Z(W) : V(\tilde{\Sigma}) \to V(\tilde{\Sigma}')$ will send $V(\tilde{\Sigma})_0$ to $V(\tilde{\Sigma}')_0$. Thus there is an induced map $V(\tilde{\Sigma})_\sharp \to V(\tilde{\Sigma}')_\sharp$ which we will denote $Z(W)_\sharp$.*

Proof. Suppose that $x \in V(\tilde{\Sigma})_0$, then for some $k > 0$, $Z(E_k)x = 0$. Since $W \cup E'_k = E_k \cup T^k W$,

$$Z(E'_k) \circ Z(W) = Z(T^k W) \circ Z(E_k). \tag{2.5}$$

It follows that $Z(E')^k(Z(W)x) = 0$. This means $Z(W)x \in V(\tilde{\Sigma}')_0$. $\square$

Lemma(2.6). *$Z(W)_\sharp : V(\tilde\Sigma)_\sharp \to V(\tilde\Sigma')_\sharp$ is injective.*

Proof. Let $x \in V(\tilde\Sigma)$, and let $[x]$ denote the the image of x in $V(\tilde\Sigma)_\sharp$. Suppose $Z(W)_\sharp([x]) = 0$. Then $Z(W)(x) \in V(\tilde\Sigma')_0$. So for some k, $Z(E'_k)((Z(W)x) = 0$. For some $l > 0$, $\tilde\Sigma_l \notin W \cup E'_k$. Let X be the compact submanifold of M_∞ with boundary $-\tilde\Sigma'_k \coprod \tilde\Sigma_l$. Thus $Z(X) \circ Z(E'_k) \circ Z(W)(x) = 0$. Since $E_l = W \cup E'_k \cup X$, $x \in V(\tilde\Sigma)_0$, thus $[x] = 0$. $\square$

By Lemma (2.6) and Equation (2.5), the hypothesis of Proposition (2.2) are now satisifed with $\mathcal{Z} = Z(E)$, and $\mathcal{Z}' = Z(E')$. Thus $\mathcal{I}(Z(E)) = \mathcal{I}(Z(E'))$. Thus $Z(E)_\natural$ is an isomorphism. Now as in the proof of Lemma (1.4), we have that the similarity class of $Z(E)_\natural$ does not depend on the choice of Σ. $\square$

Given a finite linearization (V, Z) over a Dedekind domain k as above, we may of course obtain a finite linearization, say, $(\hat V, \hat Z)$ over $\hat k$, the field of fractions of k, by tensoring with $\hat k$. Then we have $\mathfrak{A}_Z(M,\chi) \otimes \hat k = \mathcal{A}_{\hat Z}(M,\chi)$ and $\mathfrak{M}_Z(M,\chi) \otimes \hat k = \mathcal{M}_{\hat Z}(M,\chi)$. We also define $\Gamma_Z(M,\chi) = \Gamma_{\hat Z}(M,\chi)$ and $D_Z(M,\chi) = D_{\hat Z}(M,\chi)$. Using (2.1), we have:

Proposition (2.7). *Suppose there is a surface Σ dual to χ such that $V(\Sigma)$ is a free k module and $Z(E(\Sigma))$ is injective, then $\mathcal{I}(M,\chi)$ is the principle ideal generated by $D_Z(M,\chi)$.*

The hypothesis of (2.7) usually holds in the examples we have studied. However it does not always hold.

§3 The (V_p, Z_p) Theories

From now on, we will mainly be discussing the (V_p, Z_p) theories of [BHMV1]. These are defined on the cobordism category $C_2^{p_1}$ and on the larger category $C_{2,q}^{p_1,c}$ [BHMV1,4.6] whose objects are surfaces with p_1-structure with q-colored banded points, and whose morphisms are 3-manifolds with p_1-structure with a banded trivalent q-colored graph with admissible q-coloring. Here a q-coloring assigns to each edge or framed point an integer from zero to $q-1$. Here q is $\frac{p-2}{2}$ if $p \geq 4$ and is even, and is $p-1$ if $p \geq 3$ and is odd. In the case p is one of two, we take q to equal two and assume that the union of the edges of the graph weighted one is a link. We will call one of these integers a q-color. A good q-color is simply a q-color if p is even and is an even q-color if p is odd. Recall [BHMV1] a triple of q-colors (i,j,k) is called admissible if $i+j+k \equiv 0 \pmod 2$, and $i \leq j+k$, $j \leq i+k$, and $k \leq i+k$. We will say that an admissible triple (i,j,k) is small if in addition $i+j+k < 2q$. A coloring is said to be admissible if the colors of the edges meeting at any vertex of order three form an admissible triple. A coloring is small if these admissible triples are small. From now on we will refer to an admissibly q-colored trivalent banded graph as simply a colored graph. Note that the notion of a colored graph includes the notion of a colored link, and plain banded link as special cases. A colored graph which happens to be a link (i.e. there are no 3-valent vertices) will be called a colored link.

One may regard $C_2^{p_1}$ as a subcategory of $C_{2,q}^{p_1,c}$ by assigning one uniformly. The target category for (V_p, Z_p) is the category of free finitely generated k_p modules, and module homomorphisms. The functor from $C_2^{p_1}$ is the composition of inclusion and the functor from $C_{2,q}^{p_1,c}$. Thus we do not really need to distinguish between these

two linearizations in our notation. If the structure of M contains a banded link (colored graph) we will denote it by L (G).

We say L is an even link in (M, χ) if χ reduced modulo two is trivial on the nonoriented fundamental class of L. Otherwise L is an odd link in (M, χ). A colored graph is odd or even according to whether its expansion [BHMV1] is.

Proposition 3.1. *If the colored graph in (M, χ) is odd, $\mathfrak{M}_{Z_p}(M, \chi) = 0$.*

Proof. In this case $V_p(\Sigma)$ is zero as a surface with an odd number of points is not a boundary. $\square$

Proposition (3.2). *$\mathfrak{A}_{Z_p}(M, \chi)$ does not change if we vary the p_1-structure on M by a homotopy.*

Proof. Suppose we change the p_1-structure on M by a homotopy, then we change the p_1-structure on E by a homotopy during which the p_1-structure induced on the two copies of Σ is identical. Let E' denote E with the new p_1-structure, and let Σ' denote Σ with the new p_1-structure. We use this homotopy of p_1-structure restricted to Σ to put a p_1-structure on $I \times \Sigma$. As $L \cap \Sigma$ or $G \cap \Sigma$ defines some framed points in Σ, $(-1, 1)$ times these framed points defines a banded link in $I \times \Sigma$. Let P denote $I \times \Sigma$ equipped with the above p_1-structure, and banded link. Let $E'' = P \cup E \cup -P$ glued along the two copies of Σ. E' and E'' both define morphisms from Σ' to Σ'. There is a diffeomorphism from E' to E'', and if we pull back the p_1-structure on E'' to E', it is homotopic to the p_1-structure on E'. Similarly if we pull back the banded link in E'' to E', it is isotopic to the link in E'. Thus our morphisms $Z_p(E'')$ to $Z_p(E')$ are equal. Clearly $Z_p(E'')$ is similar to $Z_p(E)$. $\square$

Proposition (3.3). *$\kappa^{-\sigma(\alpha(M))}\mathfrak{A}_{Z_p}(M, \chi)$ is invariant as we vary the p_1-structure on M.*

Proof. If we change the homotopy class of the p_1-structure we may do that in a small ball neighborhood well away from Σ. This lifts to a change in the p_1-structure of E which takes place in a small ball in the interior of E. Using the functorial properties of (Z_p, V_p), this will change $Z_p(E)$ and $Z_p(M)$ by the same nonzero factor. By [BHMV1,(1.8)], this factor is compensated for by $\kappa^{-\sigma(\alpha)}$. $\square$

In view of the above, we let $Z_p(M, \chi)$ denote $\kappa^{-\sigma(\alpha)}\mathfrak{A}_{Z_p}(M, \chi)$. We also let $\hat{Z}_p(M, \chi)$ denote $\kappa^{-\sigma(\alpha)}\mathcal{A}_{\hat{Z}_p}(M, \chi)$. In this way, we may remove the dependence of our invariants on the p_1-structure. The dependence on the banding of the link L or colored graph G is similar. However there is no integer invariant of this banding that can be defined in this generality to play the role of σ. Since $e_i \in V_p(S^1 \times S^1)$ is an eigenvector for the twist map with eigenvalue $\mu(s) = (-1)^s A^{s^2+2s}$ [BHVM1,(5.8)], we can show

Proposition (3.4). *$Z_p(M, \chi)$ is multiplied by $\mu(s)$ when we change the banding on the colored graph G by adding a single positive full twist to a single edge colored s.*

The behavior of the σ-invariant of p_1-structures under a cover is related to signature defects [H] [KM2]:

Proposition (3.5). *Given (M, χ) with p_1-structure $\alpha(M)$ and $\chi \in H^1(M)$, we have $\sigma(\alpha(M_d)) = d\sigma(\alpha(M)) - 3$ def (M, χ_d).*

If N is a morphism from $\emptyset$ to $\emptyset$ in $C_2^{p_1}$ or $C_{2,q}^{p_1,c}$, we follow [BHMV1] and denote $Z_p(N) \in k_p$ by $< N >_p$.

Let $\mathfrak{i}$ denote an embedding of k_p in $\mathbb{C}$ which sends A to $e^{\frac{\pi i}{p}}$ and sends η to a positive number. There is such an embedding since $\mathfrak{i}(\kappa^3)$ is only determined up to sign by the choice of $\mathfrak{i}(A)$. Then $[n] = \frac{A^{2n}-A^{-2n}}{A^2-A^{-2}}$ will be sent to a positive number for $n < p$. If (i_1, i_2, i_3) is a small admissible triple of q-colors, with associated internal colors (α, β, γ) then $\mathfrak{i}([k]) > 0$ if k is one of i_1, i_2, i_3, α, β, γ, $\alpha+\beta+\gamma+1$. Thus, in this situation, $(-1)^{\alpha+\beta+\gamma}\mathfrak{i}(< i_1, i_2, i_3 >) > 0$. Also for any q-color c, $\mathfrak{i}([c]) > 0$. [BHMV1,(4.11),(4.14)] describes a basis for $V_p(\Sigma)$ given by a small admissible coloring of a trivalent graph in a handlebody with boundary Σ. They also describe the Hermitian form $< >_\Sigma$ on $V_p(\Sigma)$.

Proposition (3.6). *If we extend our coefficients to $\mathbb{C}$ by $\mathfrak{i}$, then the form $< >_\Sigma$ on $V_p(\Sigma) \otimes \mathbb{C}$ is positive definite.*

By (1.6) we have:

Corollary (3.7). *Suppose M is a fiber bundle over a circle with fiber Σ and let $\chi \in H^1(M)$ be the cohomology class which is classified by the projection. Then the roots of $\mathfrak{i}(\Gamma_p(M, \chi))$ lie on the unit circle.*

By the triangle inequality and (1.7), we have:

Corollary (3.8). *Suppose M is a fiber bundle over a circle with fiber Σ and let $\chi \in H^1(M)$ be the cohomology class which is classified by the projection. For all d, $|\mathfrak{i}(< (M, \chi)_d >_p)| \le \dim V_p(\Sigma)$.*

Proposition (3.9). *Let Σ denote a surface and suppose $\Sigma \times S^1$ is given a p_1-structure with σ zero. For $p \ge 3$, $< \Sigma \times S^1 >_p = \mathrm{rank}_{k_p} V_p(\Sigma)$.*

Proof. Give Σ a p_1-structure, give S^1 the p_1-structure coming from a framing on S^1. The mapping torus of the identity map on Σ, is $\Sigma \times S^1$ with the product p_1-structure which we denote by α. $\Sigma \times S^1$ is the boundary of $\Sigma \times D^2$, and the p_1-structure on $\Sigma \times S^1$ extends over this 4-manifold as any p_1-structure on S^1 extends to one on D^2. Since the signature of $\Sigma \times D^2$ is zero, $\sigma(\alpha)$ is zero. Also $\mathrm{rank}_{k_p} V_p(\Sigma)$ is the trace of the identity on $V_p(\Sigma)$. $\square$

Of course the above proposition is well known except possibly for nailing down the σ invariant.

Corollary (3.10). *Suppose M is a fiber bundle over a circle with fiber Σ with mondromy of period s. Suppose the colored graph in M is empty. Let $\chi \in H^1(M)$ be the cohomology class which is classified by the projection. Assume $p \ge 3$. If $p \equiv 0 \pmod 4$ or $p \equiv -1 \pmod 4$, then $Z_p(M, \chi)$ is a periodic map with period $2ps$. If $p \equiv 2 \pmod 4$ or $p \equiv 1 \pmod 4$, then $Z_p(M, \chi)$ is a periodic map with period $4ps$.*

Proof. Let Σ be the fiber. Let E be the associated fundamental domain of the infinite cyclic cover of M and $E_s = \cup_{0 \le i \le s-1} T^i E$ as in §1. E_s is diffeomorphic to $\Sigma \times [0, 1]$, forgetting p_1-structure . Thus $(Z_p(E))^s = Z_p(E_s)$ is then given by a scalar multiple, say c, of the identity. By (3.5), M_s has an induced p_1-structure with $\sigma(\alpha(M_s)) = -3$ def (M, χ_s). $< M_s >_p = c \dim V_p(\Sigma)$, the trace of $Z_p(E_s)$.

Whereas if we gave $\Sigma\times[0,1]$ the product p_1-structure, then $Z_p(\Sigma\times[0,1])$ would be the identity, and by (3.9), the associated mapping torus, with a p_1-structure with σ equal to zero, would have $<\ >_p$ equal to $\dim V_p(\Sigma)$. Thus $c=\kappa^{-9\operatorname{def}\,(M,\chi_s)}$ is a power of κ^3. If $p\equiv 0 \pmod 4$ or $p\equiv -1 \pmod 4$, this is a $2p$th root of unity. If $p\equiv 1 \pmod 4$ or $p\equiv 2 \pmod 4$, this is a $4p$th root of unity. So in the first case $(Z_p(E))^{2ps}$ is the identity. In the second case $(Z_p(E))^{4ps}$ is the identity. $\square$

§4 Links in $S^1\times S^2$ and their wrapping numbers

Now we consider $\hat{Z}_p(M,\chi)$ where $M=S^1\times S^2$ containing a banded link L, and χ evaluates to one on the S^1 factor. We let $\hat{z}_p(L)$ denote this invariant. We will also let $z_p(L)$ denote $Z_p(M,\chi)$. It turns out that for almost all p, $\hat{z}_p(L)$ is actually the reduction of a single automorphism of a free $\mathbb{Q}[A,A^{-1}]$-module. To see this we first define a finite linearization over $\mathbb{Z}[A,A^{-1}]$ of a certain weak cobordism category $\mathcal{C}$. By a weak cobordism category, we mean a cobordism category which does not have a disjoint union operation. The involution on the ring $\mathbb{Q}[A,A^{-1}]$ and $\mathbb{Z}[A,A^{-1}]$ sends A to A^{-1} and fixes $\mathbb{Q}$.

We form $\mathcal{C}$ by taking the definition of the category $C_2^{p_1}$, throwing out any mention of p_1 structure, insisting that every object be diffeomorphic to either S^2 with an even number of banded points or the $\emptyset$, and insisting that every morphism be diffeomorphic to either S^3, D^3, $S^2\times I$ or $\emptyset$ equipped with a banded link which meets each boundary component in an even number of points.

Now the Kauffman bracket on banded links in S^3, is an involutory $\mathbb{Z}[A,A^{-1}]$-valued invariant of closed objects of $\mathcal{C}$ and so defines a linearization $(\mathcal{V},\mathcal{Z})$ over $\mathbb{Z}[A,A^{-1}]$. If Σ is a nonempty object with $2n$ banded points then $\mathcal{V}(\Sigma)$ is the Kauffman skein module of $(B,2n)$, where B is a 3-ball with boundary Σ. This module may be identified with K_n, the Kauffman skein module for the disk with $2n$-boundary components discussed by Lickorish [L2]. This has a basis $\{D_i\}$ consisting of all the isotopy classes of configurations of n arcs in D^2 with boundary the collection of $2n$ points with diagrams with no crossings. The number of such diagrams and thus the dimension of this $\mathbb{Z}[A,A^{-1}]$-module is the nth Catalan number $c(n)=\frac{1}{n+1}\binom{2n}{n}$. One also has $\mathcal{V}(\emptyset)=\mathbb{Z}[A,A^{-1}]$. Thus $\mathcal{V}(\Sigma)$ will be free and has finite rank.

If $M=S^1\times S^2$ containing an even banded link L, and χ evaluates to one on the S^1 factor, then in the construction of §1 we may always take Σ to be an object of $\mathcal{C}$, and $E(\Sigma)$ to be a morphism of $\mathcal{C}$. In this way we obtain an endomorphism $\mathcal{Z}(E)$ of $\mathcal{V}(\Sigma)$. The results (1.1)-(1.5), (2.3)-(2.6) apply just as well to finite linearization of a weak cobordism category.

Given a link L in $S^1\times S^2$, we may isotope it so that it lies in $S^1\times B^2$, is transverse to $\{1\}\times S^2$ and has a regular projection to $S^1\times B^1$. Next we cut the diagram of the projection to $S^1\times B^1$ along $\{1\}\times B^1$, to obtain a link diagram $\mathcal{T}$ in $I\times B^1$. By the number of strands of $\mathcal{T}$, we mean the number of points in $\mathcal{T}\cap\{0\}\times B^1$. Suppose for now that $\mathcal{T}$ has $2n$ strands. Let $\mathcal{D}_i$ be a diagram for D_i in $[-1,0]\times B^1$ with the $2n$ points on $\{0\}\times B^1$. Let $\mathcal{Q}(\mathcal{T})$ be the $c(n)\times c(n)$ matrix over $\mathbb{Z}[A,A^-1]$, whose (i,j) entry is given by the coefficient of the skein element D_j when $D_i\cup\mathcal{T}$ in $[-1,1]\times B^1$ is written in terms of the basis $\{D_j\}$ with the $2n$ points now on $\{1\}\times B^1$. Let $\Gamma(\mathcal{T})\in\mathbb{Z}[A,A^{-1}]$ denote the normalized characteristic polynomial of $\mathcal{Q}(\mathcal{T})$.

There is an alternative to the above method method for writing out $\mathcal{Q}(\mathcal{T})$. Let

$D(n)$ denote the $c(n) \times c(n)$ matrix whose (i,j) entry is the bracket of the diagram in S^2 obtained by taking the union of the diagram for D_i with the diagram for D_j along their boundary. As this is a diagram without crossings this entry is just $\delta = -(A^2 + A^{-2})$ raised to the number of components in the resulting diagram in S^2. This matrix was first considered by Lickorish [L2]. $\det D(n)$ is a nonzero polynomial in δ. It is easy to see that the diagonal entries of $D(n)$ are just δ^n and the off diagonal entries are δ to smaller powers. Thus the degree of $\det D(n)$ in δ is $nc(n) = \binom{2n}{n}$. Thus $D(n)$ is invertible over the field of rational functions $\mathbb{Q}(A)$. Let $B(\mathcal{T})$ be the matrix over $\mathbb{Z}[A]$, whose entries are given by the bracket polynomial of the diagram $D_i \cup \mathcal{T} \cup m(D_i)$ in $[-1,2] \times B^1$. Here $m(D_i)$ is the diagram in $[1,2] \times B^1$ obtained by reflecting D_i across the line $\{1/2\} \times B^1$. Note if $\mathcal{T}$ is a diagram consisting of $2n$ straight strands, then $B(\mathcal{T}) = D(n)$. In general, we have $\mathcal{Q}(\mathcal{T}) = B(\mathcal{T})D(n)^{-1}$. Note that this method of calculating $\mathcal{Q}(\mathcal{T})$ does not make it apparent that the entries of $\mathcal{Q}(\mathcal{T})$ lie in $\mathbb{Z}[A, A^{-1}]$.

We now consider the quantization $(\hat{\mathcal{V}}, \hat{\mathcal{Z}})$ over the rational functions $\mathbb{Q}(A)$. Let $\mathcal{Q}(\mathcal{T})_\flat$ denote the induced automorphism of $(K_n \otimes \mathbb{Q}(A))_\flat$, as in §1. $\Gamma(\mathcal{T})$ is the characteristic polynomial of $\mathcal{Q}(\mathcal{T})_\flat$. Thus we have:

Theorem (4.1). *The similarity class of $\mathcal{Q}(\mathcal{T})_\flat$, and the polynomial $\Gamma(\mathcal{T})$ are invariants of L.*

Thus we may let $\Gamma(L)$ denote $\Gamma(\mathcal{T})$, and let $D(L)$ denote the constant term of $\Gamma(\mathcal{T})$. The wrapping number $w(L)$ is the minimum number of transverse intersections of L with an essential embedded 2-sphere [Li].

Corollary (4.2). *If L is an even link in $S^1 \times S^2$ with diagram $\mathcal{T}$ with $2n$ strands where $n \geq 2$ and $\det(B(\mathcal{T}))$ is nonzero, then $w(L) = 2n$.*

Proofs. If $\det(B(\mathcal{T}))$ is nonzero, then $\det(B(\mathcal{T})D(n)^{-1})$ is nonzero. Thus $\Gamma(L)$ has degree $c(n)$. Since $m < n$ implies $c(m) < c(n)$ for $n > 2$, the conclusion follows. □

Corollary (4.3). *If L is an even link in $S^1 \times S^2$, then $c(\frac{w(L)}{2}) \geq deg(\mathcal{G}(L))$.*

Proof. We may calculate $\mathcal{G}(L)$ from a tangle with $w(L)$ of strands. □

Hoste and Przytycki have calculated the Kauffman skein module of $S^1 \times S^2$ and have in this way obtained results on the wrapping number [HP]. Our results appear to be different but a detailed comparison has not been done. Hoste and Przytycki show the Kauffman skein module modulo $\mathbb{Z}[A, A^{-1}]$-torison is $\mathbb{Z}[A, A^{-1}]$, and let π denote the quotient map. In [Gi], we show that the trace of $\mathcal{Q}(\mathcal{T})$ is the same as $\pi(L)$.

Now we consider an intermediate linearization $(\check{\mathcal{V}}, \check{\mathcal{Z}})$ over the PID $\mathbb{Q}[A, A^{-1}]$. Let $\mathcal{Q}(\mathcal{T})_\sharp$ denote the induced 1-1 endomorphism of $(K_n \otimes \mathbb{Q}[A, A^{-1}])_\sharp$, as in §2. $\Gamma(\mathcal{T})$ is its characteristic polynomial. Thus $\mathcal{I}_{\check{\mathcal{Z}}}(M, \chi)$ is the principle ideal generated by $D(L)$. Thus in this case $k_{\mathcal{I}}$ is $\mathbb{Q}[A, A^{-1}, \frac{1}{D(L)}]$. Let $\mathcal{Q}(\mathcal{T})_\natural$ denote the induced automorphism of $K_n \otimes \mathbb{Q}[A, A^{-1}, \frac{1}{D(L)}]$ as in §2. We have :

Theorem (4.4). *The similarity class of $\mathcal{Q}(\mathcal{T})_\natural$, over $\mathbb{Q}[A, A^{-1}, \frac{1}{D(L)}]$ is an invariant of L.*

So we may let $z(L)$ denote the similarity class of $\mathcal{Q}(\mathcal{T})_\natural$, over $\mathbb{Q}[A, A^{-1}, \frac{1}{D(L)}]$.

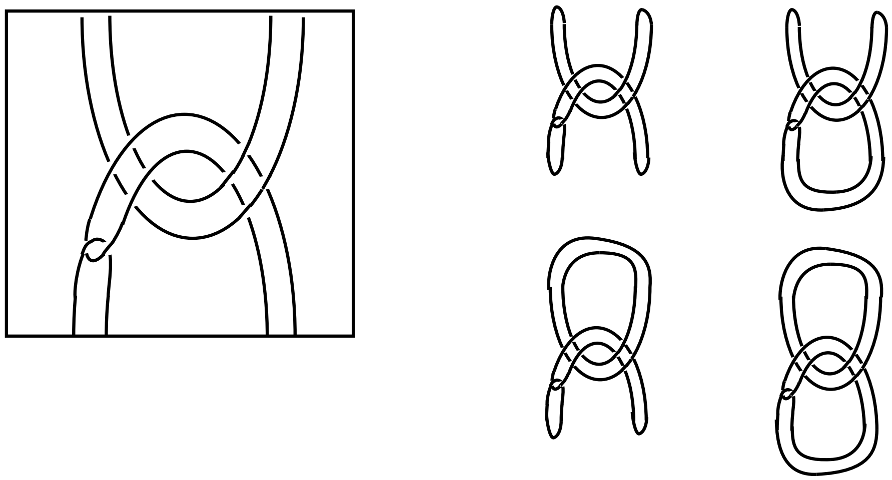

Figure 1

Example (4.5) Let $\mathcal{T}$ be the tangle on the left of Figure 1. It can be closed up to form a link L in $S^1 \times S^2$. $B(\mathcal{T})$ is given by taking the bracket polynomial of the matrix of link diagrams on the right of Figure 1. Thus $B(\mathcal{T}) = \begin{bmatrix} \delta h & \delta^2 A^6 \\ \delta^2 h & w \end{bmatrix}$, where $h = -\delta(A^4 + A^{-4})$, the bracket of the standard diagram for the Hopf link, and

$$w = -2 + A^{-16} - A^{-8} - A^{-4} - 2A^4 - 2A^8 - A^{20}.$$

Kauffman has a good method for calculating the bracket of link diagrams with double strands [KL,(§4.4)]. See also [K2], an early version of [K3]. We call this the Kauffman double bracket method. We used this method to calculate w. Thus $\mathcal{Q}(\mathcal{T})$ is given by

$$\begin{bmatrix} -1 - A^{-4} & -A^{-2} + A^6 \\ A^{-10} - A^{-6} + 3A^{-2} + A^2 - A^6 + 2A^{10} - A^{14} & A^{-12} - A^{-8} + 2 - 2A^4 + A^{12} - A^{16} \end{bmatrix}$$

This matrix has a nonzero determinant

$$D(L) = -A^{-16} + A^{-12} + 2 - 2A^4 - A^{16} + A^{20}.$$

Thus the above matrix represents $z(L)$, and $\Gamma(L) = g_0 + g_1 x + x^2$ where $g_0 = D(L)$ above and

$$g_1 = -A^{-12} + A^{-8} + A^{-4} - 1 + 2A^4 - A^{12} + A^{16}.$$

We may conclude that the wrapping number of L is four. This also follows from [Li], as well as [HP]. Using (1.5) and an analog of (3.4), we conclude that L is not isotopic to its image under a orientation reversing diffeomorphism of $S^1 \times S^2$ ignoring banding.

If X is either a scalar in $\mathbb{Z}[A]$ or a matrix over $\mathbb{Z}[A]$, then we let either ${}_pX$ or X_p (depending on where there is more room for the subscript) denote X after evaluating at $A = A_p$. Similarly if X is a module or module homomorphism over $\mathbb{Z}[A, A^-1]$ or $\mathbb{Q}[A, A^{-1}]$ let X_p denote the result of tensor product with k_p or Id_{k_p}.

We will say p *is ordinary with respect to* n if and only if $D(n)_p$ is nonsingular. If p is not ordinary, we will say it is *special with respect to* n. Ko and Smolinsky

[KS] studied the question: when is p ordinary with respect to n? We note first that since $\det D(n)$ is a nonconstant polynomial almost all p are ordinary with respect to a given n.

Ko and Smolinsky showed that all the roots of $D(n)_p$ are of the form $\delta = 2\cos(\frac{k\pi}{m+1})$ where $1 \le k \le m \le n$. We note that one and two are ordinary with respect to any n. Ko and Smolinsky showed that $2r$ is special with respect to $r-1$, as required by Lickorish [L2].

By [BHMV1,(1.9)], there is an epimorphism $\epsilon_{(D^3,2n)} : K(D^3,2n)_p \to V_p(S^2,2n)$. Here we let (S^2,m) denote the 2-sphere with m framed points. Using the non-singular Hermitian form on $V_p(S^2,2n)$, one sees that $\epsilon_{(D^3,2n)}$ is an isomorphism if and only if p is ordinary with respect to n. In this case $\{D_i\}$ describes a basis for $V_p(S^2,2n)$ with cardinality $c(n)$. [BHMV1,(4.11 & 4.14)] gives bases for $V_p(S^2,2n)$ and so may also be used to determine when $\epsilon_{(D^3,2n)}$ is an isomorphism. We obtain:

Proposition (4.6). *If $p \ge 4$ is even, then p is ordinary with respect to n if and only if $p > 2n+2$. If $p \ge 3$ is odd, then p is ordinary with respect to n if and only if $p > n+1$.*

Theorem (4.7). *If p is ordinary with respect to n, and $D(L)_p \ne 0$, then $\Gamma_p(L) = (\Gamma(L))_p$, and $\hat{z}_p(L) = z(L)_p$*

Proof. If p is ordinary with respect to n, $\hat{Z}_p(I \times S^2, \mathcal{T})$ with respect to the basis $\{D_i\}$ is represented by the matrix $\mathcal{Q}(\mathcal{T})_p$. If $D(L)_p \ne 0$, then $(\mathcal{Q}(\mathcal{T})_\natural)_p$ represents $\hat{Z}_p(I \times S^2, \mathcal{T})_\flat$. $\square$

We note that the hypothesis of (4.7) is true for almost all p if $\Gamma(L) \ne 0$. Although it would be interesting to calculate $z_p(L)$, or $\hat{z}_p(L)$ for p which are special with respect to n, we do not pursue this now. In order to define invariants for odd links in $S^1 \times S^2$ we have several options. One could color the link L with a fixed even integer, say two, and evaluate the above invariants in the colored theories. Alternatively one could consider the link obtained by replacing each component of L with two strands using the banding to form a new banded link L', and then calculate the invariants of the even link L'. Note that L' is formed by taking a pair of "scissors" and splitting each band in L to form two bands. One could use other "even" satellite constructions. In §10 we will mention a third method of obtaining invariants of odd links.

§5 Knot invariants

Given an oriented knot K in a homology sphere S, we may let $S(K)$ denote zero framed surgery to S along K. $S(K)$ has the integral homology of $S^1 \times S^2$. We let χ denote the cohomology class which evaluates to be one on a positive meridian of K. Let $Z_p(K)$ denote $Z_p(S(K),\chi)$, $\hat{Z}_p(K)$ denote $\hat{Z}_p(S(K),\chi)$, and $\Gamma_p(K) = \Gamma_p(S(K),\chi)$. We will also let $\mathcal{E}(K)$ denote the list of eigenvalues of $\hat{Z}_p(K)$ counted with multiplicity. By Proposition (1.5), one can see that these invariants do not depend on the string orientation of the knot. Let $-K$ denote the knot obtained by taking the mirror image of K and reversing the string orientation. This knot represents the inverse of K in the knot cobordism group. Then $Z_p(-K) = (Z_p(K))^*$. Let U denote the unknot in S^3, then $S^3(U) = S^1 \times S^2$. So far for the examples we have calculated $\hat{Z}_p(K)$ is diagonalizable. It would be interesting to

find two knots K_1 and K_2 such that $\Gamma_p(K_1) = \Gamma_p(K_2)$, but $\hat{Z}_p(K_1) \neq \hat{Z}_p(K_2)$, or $Z_p(K_1) \neq Z_p(K_2)$.

Theorem (5.1). *If K is a fibered knot in a homology sphere which is a homotopy ribbon knot, then one is a root of $\Gamma_p(K)$.*

Proof. According to Casson and Gordon [CG1], a fibered knot in a homology sphere is homotopy ribbon if and only if the the closed monodromy extends over a handlebody H. In this case the ordinary mapping torus R of the extension to the handlebody is a homology $S^1 \times B^3$ which embeds naturally in a homology ball B with boundary S in which K bounds a homotopy ribbon disk Δ. In fact R is basically the exterior of Δ. We can give B a p_1-structure.. It will induce a p_1-structure on R which in turn induces a p_1-structure on $S(K)$ with σ zero. Note H represents an element in $V(\partial H) = V(\Sigma)$ which is fixed by T. Thus H represents an eigenvector with eigenvalue one. $\square$

Remark Although there is an algorithm to answer the question of whether a given diffeomorphism extends over a handlebody [CL], consideration of the eigenvalues of a map induced under a TQFT functor may turn out to be a good way to show that a diffeomorphism does not extend over a handlebody or even bound in the bordism group of diffeomorphisms [B],[EW].

The following proposition follows instantly from [BHMV1,(1.5)]. Here $i_p : k_2 \to k_{2p}$ and $j_p : k_p \to k_{2p}$ are the homomorphisms defined in [BHMV1]. We must note that for p odd, $i_p(\kappa_2)j_p(\kappa_p) = \kappa_{2p}$. We use the same symbols to describe the induced maps on similarity classes and polynomials over these rings. We will discuss the tensor product of polynomials in the Appendix.

Proposition (5.2). *If p is odd, then $Z_{2p}(K) = i_p(Z_2(K)) \otimes j_p(Z_p(K))$, and so $\Gamma_{2p}(K) = i_p(\Gamma_2(K)) \otimes j_p(\Gamma_p(K))$.*

Let U denote the unknot. We will say the similarity class of the identity on a free module of rank one is trivial.

Theorem (5.3). *For all p, $Z_p(U)$ is trivial. If p is one, three or four, and K is a knot in S^3, then $Z_p(U)$ is also trivial.*

Proof. The first statement follows directly from the definitions. If p is one, three or four, we have [BHMV1,§2][BHMV2,§6] $\omega = \eta$, $\kappa^{-3}\eta = 1$, $\kappa^6 = 1$, and thus $\kappa^3\eta = 1$. One may obtain any knot in S^3 from the unknot U by doing ± 1 surgery to the components of an unlink in the complement of U where the linking number of each component with U is trivial [R,(6D)]. Thus we may pick a Seifert surface F for U in the complement of this unlink. We may calculate $Z_p(U)$ from an E that we construct with $\Sigma = F$ capped off. Here we give S^3 a p_1-structure with σ zero. Let us calculate the effect of a single surgery on E. Let E' denote the result of performing $+1$ framed p_1-surgery to E. Then $Z_p(E') = \omega_p Z_p(E) = \eta Z_p(E)$. Let K' denote the image of U after this surgery. Then $Z_p(K') = \kappa^{-3} Z_p(E') = Z_p(E) = Z_p(U)$. If we were to perform -1 surgery then the invariant would change by a factor of $\kappa^3\eta = 1$. This same argument may be repeated to show the further surgeries do not change $Z_p(K)$. $\square$

In order to obtain (5.6) below about periodicity of $Z_p(K)$, for K fibered with periodic monodromy, we make the following definitions and observations which will be useful later as well. Let $\sigma_\omega(K) = \mathrm{Sign}((1-\omega)V + (1-\bar{\omega})V^t)$, where $\omega \in \mathbb{C}$

with $|\omega| = 1$ and V is a Seifert matrix for K. Following [KM2], let $\sigma_d(K) = \sum_{i=1}^{d-1} \sigma_{\omega_d^i}(K)$, where $\omega_d = e^{2\pi i/d}$. These are called the total d-signatures of K. Our convention is that $\sigma_1(K)$ is taken to be zero. Also $\operatorname{def}(S^3(K), \chi_d) = -\sigma_d(K)$. By (3.5) we have:

Proposition (5.4). *Let $\alpha(K,d)$ be the p_1-structure on $S(K)_d$ induced from $S(K)$, $\sigma(\alpha(K,d)) = 3\sigma_d(K)$.*

Lemma(5.5). *If K is a fibered knot with a periodic monodromy of order s, then $\sigma_{s+1}(K) \equiv 0 \pmod 8$, and $\sigma_s(K) \equiv 0 \pmod 8$.*

Proof. Let D_d denote the d-fold cyclic branched cover of D^4 along a pushed in Seifert surface for K. D_d is a spin simply connected manifold with boundary K_d and $\operatorname{Sign}(D_d) = \sigma_d(K)$. K_{s+1} is the result of $1/n$ surgery on K for some n. Thus K_{s+1} is a homology sphere. So the signature of D_{s+1} is the signature of an even unimodular symmetric matrix and so is zero modulo eight. $(S^3(K))_s$ is diffeomorphic to $\Sigma \times S^1$. It follows that $H_1(K_s)$ is torsion free. Thus the induced form on $H_2(D_s)$ modulo the radical of the intersection form is given by an even unimodular symmetric matrix and so has signature zero modulo eight. $\square$

Proposition (5.6). *Suppose K is a fibered knot with a periodic monodromy of order s. Let $u^{4(\sigma_s(K)/8)}$ have order h in k_p. Note h divides p. Then $Z_p(K)$ is a periodic map with period hs.*

Proof. By(5.4), $\kappa^{\sigma(\alpha(K,d))} = u^{4(\sigma_s(K)/8)}$. The result follows from the proof of (3.10). $\square$

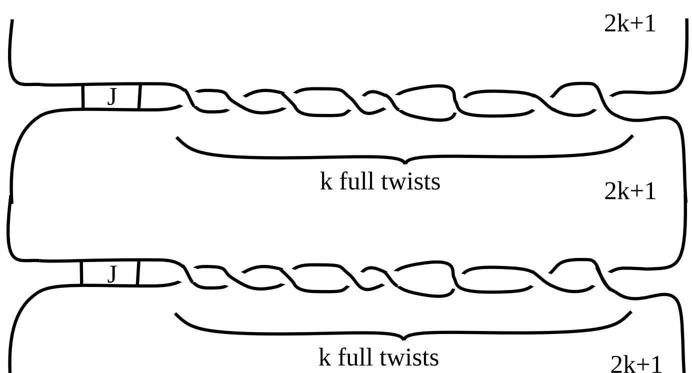

Figure 2

Given a surgery description of a knot as in Rolfsen [R,159] where each surgery curve has zero linking number with the unknot, we will give a procedure to calculate $Z_p(K)$. We use Rolfsen's method of giving a surgery description for the infinite cyclic branched cover of S^3 along K from a surgery description for K [R, p162, p158]. The same picture and argument shows that we are obtaining a description of the infinite cyclic unbranched cover of $M(K)$ as obtained by surgery to $\mathbb{R} \times S^2$. Since we are actually working with manifolds with a p_1-structure our surgery is p_1-surgery, but when we do our initial surgery to tie up K we change the p_1-structure on S^3 and $S^1 \times S^2$. In other words our surgery description describes $M(K)$ with the p_1-structure with $\frac{\sigma}{3}$ equal to the number of plus one surgeries minus the number of minus one surgeries done [BHMV1, Appendix II]. We have $Z_p(K) = \kappa_p^{-\sigma} Z_P(E'(\Sigma))$, where $E'(\Sigma)$ is the manifold we describe below.

Twisted Doubles of Knots.

We consider first the case that K is $D_k(J)$, the k-twisted double of a knot J [R,p112]. We perform a single minus one surgery to undo the clasp. See [CG2], [K1,Chapt 18] where the (finite) cyclic covers are calculated for J the unknot, and [R] for $k = 0$ and J equal to the right handed trefoil. Figure 2 gives a surgery description for the infinite cyclic cover (with $k = 4$). The box labelled J represents two parallel copies of a string diagram for J with zero writhe. To obtain the infinite cyclic cover one should perform framed surgery to $\mathbb{R} \times S^2$ along the indicated infinite chain. Consider a 2- sphere given by $\{a\} \times S^2$ which meets one component of our diagram in two points. Delete these two points and close the surface off by adding a tubular neighborhood of an arc on the intersected component. Call the resulting torus Σ.

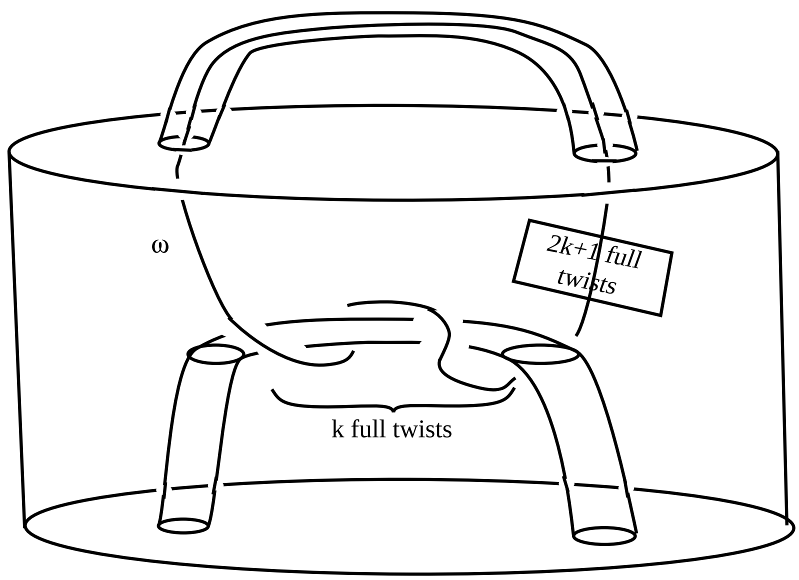

Figure 3

We can now construct $E'(\Sigma)$ by taking a “slab” $I \times S^2$, drilling out a tunnel along an arc which meets $\{0\} \times S^2$ (the arc is the top of one of the surgery curves), adding a 1-handle added along $\{1\} \times S^2$, and finally performing framed surgery along a simple closed curve γ which travels once over the one handle. According to [BHMV1,(1.C),§2,(5.8)], $Z_p(E'(\Sigma)) = Z_p(E''(\Sigma))$, where $E''(\Sigma)$ is formed by placing a linear combination of banded links ω with $2k+1$ full twists along γ where we would have performed surgery in constructing $E'(\Sigma)$. In Figure 3, we sketch $E''(\Sigma)$ when J is the unknot. The figure actually shows the case $k = 2$. We only draw an $I \times B^2$ portion of $I \times S^2$. Given an element in $\alpha \in V(\Sigma)$, we may figure out its image under $Z_p(E''(\Sigma))$ as follows. α can be represented by a banded link in a solid genus one handlebody H with boundary Σ. If we glue this handlebody to the bottom of $E''(\Sigma)$ we will obtain a linear combination of banded links in the handlebody $H \cup E''(\Sigma)$ with boundary $T(\Sigma)$. This linear combination is formed as the union of α and ω. This linear combination represents $Z_p(E''(\Sigma))(\alpha) \in V(T(\Sigma))$.

Preparatory material on the Kauffman bracket and the Kauffman polynomial.

Let $c_k(J)$ denote the bracket of the diagram, say $\mathcal{D}_k(J)$, obtained from a diagram of a knot J with writhe k by replacing each arc by 2 parallel arcs. The Kauffman double bracket method is a very efficient means for calculating $c_k(J)$. In fact $c_k(J) =$

$[\mathcal{D}_k(J)]_2+1$, [KL,(p.35-36)] and $[\ \ ]_2$ satisfies a simple skein relation. We let $[[J]]$ denote $[\mathcal{D}_0(J)]_2$. Note $[[J_1\# J_2]]=\frac{1}{\delta^2-1}[[J_1]]\ [[J_2]]$. Also $c_k(J)=A^{8k}[[J]]+1$.

If $\mathcal{D}$ is a knot diagram, then the writhe of D, denoted $w(\mathcal{D})$ is well defined. This is not true of link diagrams. One may show that $<\mathcal{D}>_2=i^{w(\mathcal{D})}2$. If J is a knot, let $<J>$ denote the bracket polynomial of a diagram for J with zero writhe. Thus $<J>_2=2$. Using the relation of the bracket polynomial and the Jones polynomial, it is easy to see that $<J>$ will be a polynomial in even powers of A. It is convenient to note that [K5,(3.2)]

$$<J>=\left(\frac{a+a^{-1}}{z}-1\right)F_J(a,z)_{|a=-A^3 \text{ and } z=A+A^{-1}}$$

Here $F_J(a,z)$ denotes the Kauffman polynomial, normalized so that it is one for the unknot. This is the form that is in the tables of [K1]. As observed in [KL], $[\mathcal{D}]_2$ is the Dubrovnik version of the Kauffman polynomial [K5,§VII] of $\mathcal{D}$ evaluated at $z=A^4-A^{-4}$, and $a=A^8$. Using Lickorish [L4], we have:

$$[[J]]=-\left(\frac{a+a^{-1}}{z}-1\right)F_J(a,z)_{|a=-iA^8 \text{ and } z=i(A^4-A^{-4})}$$

Let $\mathfrak{b}_k(J)$ denote the bracket of $\mathcal{D}_0(J)$ but with k additional full twists between the two strands. One has that

$$\mathfrak{b}_k(J)=A^{-6k}c_k(J)=A^{2k}[[J]]+A^{-6k}.$$

The following proposition now follows easily from the above formulas. This proposition seems related to identities in [P] and could perhaps be proved using them.

Proposition (5.7). *${}_p\mathfrak{b}_k(J)$ is periodic in k with period p. Also ${}_2\mathfrak{b}_k(J)=(-1)^k 4$.*

Some conventions which hold for the rest of this paper.

We will let A denote A_p, unless there may be some confusion as to which p is meant. We do not specify which primitive $2p$th root of unity this is. Similarly Ω, ω, κ, δ will denote the items defined in [BHMV1], and sometimes denoted Ω_p, ω_p, κ_p, δ_p. We let $\mu=\mu(1)=-A^3$. We also give $\kappa^{-3}\eta$ the name β. We also let $\mathfrak{b}_k(J)$ denote $\mathfrak{b}_k(J)$ evaluated at A_p, and $<J>$ denote $<J>_p$

The case $p=2$.

If Σ is a torus, then $V_2(\Sigma)$ has a basis consisting of 1 and z. We have that $\omega=\eta\Omega$ where $\Omega=1+\frac{z}{2}$, and $\beta=\frac{1-A}{2}$. With respect to the basis $\{1,z\}$, $Z_2(E(\Sigma))=\kappa^{-3}Z_2(E''(\Sigma))$ is given by $\beta\begin{bmatrix}1 & <J> \\ \frac{\mu^{2k+1}<J>}{2\delta} & \frac{\mu^{2k+1}\mathfrak{b}_k(J)}{2\delta}\end{bmatrix}$. Thus $Z_2(E(\Sigma))$ is given by $\frac{1-A}{2}\begin{bmatrix}1 & 2\\ (-1)^k\frac{A}{2} & A\end{bmatrix}$.

Proposition (5.8). *Let K be the k twisted double of J. If k is even, then $Z_2(K)$ is trivial. If k is odd, $Z_2(K)=\frac{1-A}{2}\begin{bmatrix}1 & 2\\ -\frac{A}{2} & A\end{bmatrix}_{\natural}$, and $\Gamma_2(K)$ is x^2-x+1. So $\mathcal{E}_2(K)$ is $\{1\}$ if k is even and is $\{A_3,\bar{A}_3\}$ if k is odd.*

The case $p = 5$.

If Σ is a torus, then $V_2(\Sigma)$ has a basis consisting of 1 and z. We have $\omega = \eta\Omega$. $\Omega = 1 + \delta z$, and $\beta = \frac{3-A+4A^2-2A^3}{5}$. With respect to the basis $\{1, z\}$, $Z_5(E(\Sigma)) = \kappa^{-3} Z_5(E''(\Sigma))$ is given by $\beta \begin{bmatrix} 1 & < J > \\ \mu^{2k+1} < J > & \mu^{2k+1}\mathfrak{b}_k(J) \end{bmatrix}$.

By Proposition (5.7), each entry in the above matrix has period five. Thus:

Proposition (5.9). $Z_5(D_k(J)) = \left(\beta \begin{bmatrix} 1 & < J > \\ \mu^{2k+1} < J > & \mu^{2k+1}\mathfrak{b}_k(J) \end{bmatrix}\right)_\natural$. $Z_5(D_k(J))$ *is periodic in k with period five.*

We now calculate these five invariants for various knots J. First we consider $J = U$, the unknot. We can make a number of predictions a priori. Note that $D_0(U)$ is the unknot again, so $Z_5(D_{5n}(U))$ is trivial. Also note $D_1(U)$ is the figure eight knot and $D_{-1}(U)$ is right handed trefoil. Both of these knots are fibered, so $\mathcal{E}_5(D_{5n+1}(U))$ and $\mathcal{E}_5(D_{5n-1}(U))$ should consist of elements of norm one. Since the trefoil is period with period six, by (5.4), $\mathcal{E}_5(D_{5n+1}(U))$ should consist of 30th roots of unity. Also since the figure eight is amphichiral, $\overline{\mathcal{E}_5(D_{5n+1}(U))} = \mathcal{E}_5(D_{5n+1}(U))$. We obtain:

Proposition(5.10). *If $k \equiv 0 \pmod 5$, $Z_5(D_k(U))$ is trivial. If $k \not\equiv 0 \pmod 5$, $Z_5(D_k(U)) = \beta \begin{bmatrix} 1 & \delta \\ \mu^{2k+1}\delta & \mu^{2k+1}(A^{2k}(\delta^2 - 1) + A^{-6k}) \end{bmatrix}_\natural$. In particular for all integers n, we have:*

$$\begin{array}{ll}
\Gamma_5(D_{5n}(U)) = x - 1 & \mathcal{E}_5(D_{5n}(U)) = \{1\} \\
\Gamma_5(D_{5n+1}(U)) = x^2 - (A + \bar{A})x + 1 & \mathcal{E}_5(D_{5n+1}(U)) = \{A, \bar{A}\} \\
\Gamma_5(D_{5n+2}(U)) = x^2 - (1 + \bar{A})x + \bar{A} & \mathcal{E}_5(D_{5n+2}(U)) = \{1, \bar{A}\} \\
\Gamma_5(D_{5n+3}(U)) = x^2 - (1 + \bar{A}^2)x + \bar{A} & \\
\Gamma_5(D_{5n+4}(U)) = x^2 - (\bar{A})x + (\bar{A})^2 & \mathcal{E}_5(D_{5n+4}(U)) = \{A_3\bar{A}, \bar{A}_3\bar{A}\}.
\end{array}$$

Note $\mathcal{E}_5(D_{5n+4}(U))$ are primitive 15th roots of unity. We do not list the eigenvalues for the $5n + 3$ twisted doubles. Although they are easily worked out, the formulas are not enlightening. By (5.1) and (5.10), we have that the trefoil and figure eight knots are not homotopy ribbon. This a (not very deep) four dimensional result obtained from studying a TQFT in dimension 2+1. By (7.6), below we have that the granny knot is not homotopy ribbon as well.

Let RT, LT and $F8$ denote the right handed trefoil, the left handed trefoil and the figure eight knots. For $J = RT$, LT $F8$, and the square knot $RT\#LT$ and for all k, the above matrix has nonzero determinant. Below we list $\{k, \Gamma_5(D_{5n+k}(J))\}$ for $J = RT$, LT, $F8$, and $RT\#LT$.

$J = RT$

$\{0, 1 + 2\,A^2 - 2\,A^3 - \left(2 - A + 2\,A^2 - A^3\right)\,x + x^2\}$

$\{1, -A^3 - \left(2 - A^3\right)\,x + x^2\}$

$\{2, 1 + A - A^2 - \left(1 - A + A^2 - 2\,A^3\right)\,x + x^2\}$

$\{3, 1 - 2\,A - A^3 - (1 - A)\,x + x^2\}$

$\{4, -A + A^2 + A^3 - A\,x + x^2\}$

$J = LT$

$\{0, 2-2\,A-A^3+A^3\,x+x^2\}$
$\{1, -1-A+A^2-\left(1-A+A^2\right)\,x+x^2\}$
$\{2, 1+2\,A^2-A^3-(1+A)\,x+x^2\}$
$\{3, A-A^2-A^3-\left(2-A+2\,A^2-2\,A^3\right)\,x+x^2\}$
$\{4, 1-\left(2-A-A^3\right)\,x+x^2\}$
$J=F8$
$\{0, -3+2\,A-2\,A^2+3\,A^3-A^2\,x+x^2\}$
$\{1, 3+2\,A^2-A^3-\left(2+A^2\right)\,x+x^2\}$
$\{2, 1-2\,A-A^3-\left(2+A^2-2\,A^3\right)\,x+x^2\}$
$\{3, 1+2\,A^2-A^3-\left(2-2\,A+A^2-2\,A^3\right)\,x+x^2\}$
$\{4, 1-2\,A-3\,A^3+A^2\,x+x^2\}$
$J=RT\#LT$
$\{0, -6+4A-4A^2+6A^3-\left(A-A^2+2A^3\right)x+x^2\}$
$\{1, 6+A+A^2-\left(1+A+2A^2-A^3\right)x+x^2\}$
$\{2, 1-5A+A^2-2A^3-\left(4-2A+2A^2-2A^3\right)x+x^2\}$
$\{3, 2-A+5A^2-A^3-\left(1-A^2-2A^3\right)x+x^2\}$
$\{4, -A-A^2-6A^3-\left(-2A+A^2-A^3\right)x+x^2\}$

The case $p=6$.

By (5.2), $Z_6(K)=i_3(Z_2(K))\otimes j_3(Z_3(K))$. Note that $i_3(A_2)=A_6^9=-A_6^3$. As $Z_3(K)$ is trivial, and i_p fixes $\mathbb{Z}$, we have by (5.8) that:

Proposition(5.11). *$Z_6(K)=i_3(Z_2(K)))$. In particular, if k is even $Z_6(D_k(J))$ is trivial. If k is odd, $Z_6(D_k(J))=\frac{1+A^3}{2}\begin{bmatrix}1 & 2\\ \frac{A^3}{2} & -A^3\end{bmatrix}_\natural$, and $\Gamma_6(K)=x^2-x+1$. So $\mathcal{E}_6(D_k(J))$ is $\{1\}$ if k is even and is $\{A_3,\bar{A}_3\}$ if k is odd.*

The case $p=10$.

By (5.2), we have $Z_{10}(K)=i_5(Z_2(K))\otimes j_5(Z_5(K))$, and so $\Gamma_{10}(K)=i_5(\Gamma_2(K))\otimes j_5(\Gamma_5(K))$. Thus one may work out, for instance, $\Gamma_{10}(D_k(U)$. We note that $j_5(A_5)=A_{10}^6$. By (5.8) and (5.10), $i_5(\Gamma_2(D_{5n+4}(U)))=x^2-x+1$, and $j_5(\Gamma_5(D_{5n+4}(U)))=x^2+(A_{10}^4)x+A_{10}^8$.

Using A.3 from the appendix, we obtain:

$$\Gamma_{10}(D_{5n+4}(U))=x^4+(A_{10}^4)x^3-(A_{10}^2)x-A_{10}^6. \tag{5.12}$$

The general case, $p\geq 3$.

If $p\geq 3$ then $\omega=\eta\Omega$, and $\Omega=\sum_{s=0}^{n-1}<e_s>e_s$ where e_s is an eigenvector for the twist map with eigenvalue $\mu(s)=(-1)^sA^{s^2+2s}$, and $<e_s>=(-1)^s\frac{A^{2s+2}-A^{-2s-2}}{A^2-A^{-2}}$. The $\{e_i\}_{i=0}^{n-1}$ where $n=\left[\frac{p-1}{2}\right]$ form a basis for V of a torus. We form two $n\times n$ matrices $\mathsf{B}(J,k)$ and $\mathsf{L}(J)$. $\mathsf{B}(J,k)_{i,j}=\beta\sum_{s=0}^{n-1}<e_s>\mu(s)^{2k+1}b(J,k)_{i,s,j}$ where

$b(J,k)_{i,s,j}$ is the bracket polynomial of the colored banded link in Figure 4a.

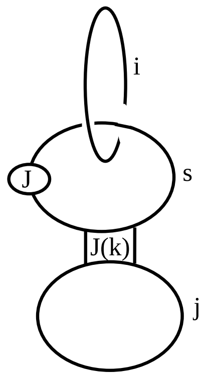

Figure 4a

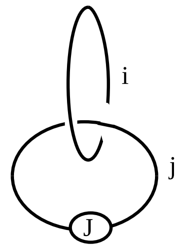

Figure 4b

The box labelled $J(k)$ represents two parallel strands of a string diagram for J with zero writhe with k additional full twists added to the strands and the circle labelled J represents a string diagram for J with zero writhe. Let $\mathsf{L}(J)_{i,j}$ be the bracket polynomial of the colored banded link in Figure 4b. For a knot J, let J_c denote J colored c.

Theorem (5.13). *If $p \geq 3$, and $< J_c >_p$ is nonzero for all $0 \leq c \leq n-1$, then $Z_p(D_k(J)) = (\mathsf{B}(J,k)\ \mathsf{L}(J)^{-1})_\natural$.*

$b(J,k)_{i,s,j}$ and $\mathsf{L}(J)_{i,j}$ may be calculated recursively using the colored Kauffman relations. These are given in [MV2] and for p even in [KL]. In fact $\mathsf{L}(U)_{i,j}$ for p even is given in [KL,p.127]. This is easily worked out for all p using the formulas in [MV2]. It is not hard to see that the summations given [MV,p 367] should be taken over k such that (i,j,k) is a small admissible triple, when one specializes to $A = A_p$. Using the colored Kauffman relations, one may deduce:

Corollary (5.14). *If $p \geq 3$, and $< J_c >_p$ is nonzero for all $0 \leq c \leq n-1$, then $Z_p(D_k(J))$ is periodic in k with period p.*

Let $\delta(k;i,j)$, and $< i,j,k >$ be as in [MV2]. We have:

$$\mathsf{L}(U)_{i,j} = \sum_{r \,\ni\, (i,r,j) \text{ is a small admissible triple}} \delta(r;i,j)^2 < r > . \tag{5.15}$$

$$b(U,k)_{i,s,j} = \sum_{\substack{r,r' \,\ni\, (i,r,s), \text{ and } (j,r',s) \\ \text{are small admissible triples}}} \delta(r;i,s)^2 \delta(r';j,s)^{2k} \frac{< r >< r' >}{< s >} \tag{5.16}$$

Of course both of these should be evaluated at $A = A_p$. We have calculated $Z_p(D_k(U))$ exactly with entries polynomials in A for $0 \leq k \leq p-1$ and for $5 \leq p \leq 16$. These as well as other lists of quantum invariants are available at gopher://math.lsu.edu. For certain knots, we have carried the calculation to higher p. We observe that for $p \leq 20$, $Z_p(RT)$ is a periodic map with period $3p$, and sometimes less. By (5.6), $Z_p(RT)$ must be periodic with period by $6p$, and sometimes less, since the trefoil has a monodromy of period six. The roots of $Z_p(F8)$ are all periodic maps for $p \leq 20$. The period is an erratic function of p. This periodicity is somewhat surprising since the monodromy for F8 is hyperbolic. We also

observe that $\Gamma_p(6_1)$ has one as an eigenvalue one for $p \leq 18$. The tweenie knot is $5_2 = -D_2(U)$ [As,p.86]. We noticed that the degree of $\Gamma_{2r}($ tweenie knot $)$ is less than $r-1 = \dim V_{2r}($ torus $)$ for $3 \leq r \leq 9$. Thus the hypothesis of (2.7) does not hold. We plan to investigate whether $\mathcal{I}(M,\chi)$ is principle in this case.

Other Knots K.

If we consider more general knots, two extra difficulties arise. First, we may not be able to calculate $Z_p(E(\Sigma))$ directly but only some power of it. As an example, starting with the surgery description of the knot 8_{16} given in [R,p.169], one may calculate $Z_p(E(\Sigma))^n$ for $n \geq 2$. Fortunately if we can calculate $Z_p(E(\Sigma))^c$, then we can also calculate $Z_p(E(\Sigma))^{c+1}$, and from this deduce $Z_p(E(\Sigma))_\natural$. Secondly, the genus of Σ may be higher than one. In the higher genus case, there is a good description [MV1],[BHMV1] of a basis of $V_p(\Sigma)$ in terms of colorings of a trivalent graph which is a deformation retract of the handlebody. Thus the same methods may be applied. The calculation becomes more difficult as p and the genus of Σ grow.

A computation with Σ genus two.

V_5 of the boundary of a genus two handlebody has a basis given by the links denoted 1, z, w, zw, and $z\#w$ in Figure 5. Starting with a surgery description of the knot 8_8, we obtain, as in Rolfsen, a surgery description of the infinite cyclic cover shown in Figure 6. Just as above, we can then find a matrix for calculating $Z_5(8_8)$. In particular, we have that $\Gamma_5(8_8) = (x-1)(x^4 + (-1-A-A^2)x^3 + (1+A^2)x^2 + (-A-A^2-A^3)x + (-1+A+A^3))$.

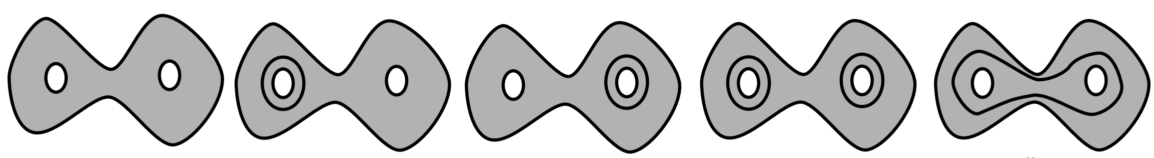

Figure 5

Figure 6

§6 Quantum Invariants of the finite cyclic covers of $S^3(K)$

For $p \geq 3$, (1.8), and the remarks following tell us how to compute $< S^3(K)_d >_p$ recursively as a function of d, once we know $\Gamma_p(K)$.

We discuss $< S^3(K)_d >_p$ for low p. We have that $< S^3(K)_d >_p = 1$, for $p = 1, 3,$ and 4. For $p = 1$, this follows easily from the definitions. For $p = 3$, and 4, this follows from (1.8) and (5.3). By (1.7) and (5.9)

(6.1) $$< S^3(D_k(J)) >_5 = \beta_5(1 + \mu_5^{2k+1}\mathfrak{b}_k(J)_5).$$

By (1.8) and (5.11)

$$(6.2)\qquad < S^3(D_k(J))_d >_6= \begin{cases} 1 & \text{if } k \text{ is even} \\ 2 & \text{if } k \text{ is odd and } d \equiv 0 \pmod 6 \\ 1 & \text{if } k \text{ is odd and } d \equiv \pm 1 \pmod 6 \\ -1 & \text{if } k \text{ is odd and } d \equiv \pm 2 \pmod 6 \\ -2 & \text{if } k \text{ is odd and } d \equiv 3 \pmod 6 \end{cases}.$$

We let $s(k,d)$ denote the function above. Again as Z_2 does not satisfy the tensor product axiom, we need an alternative method of calculating $< S^3(K)_d >_2$. $< S^3(K)_d >_2$ can be computed from $< S^3(K)_d >_6$ using [BHMV1,1.5] since $< S^3(K)_d >_3= 1$. In fact, one sees that $< S^3(K)_d >_2$ is equal to the sum of the dth powers of the roots of $\Gamma_2(K)$. Thus it turns out that $< S^3(K)_d >_p$ is the sum of the dth powers of the roots of $\Gamma_p(K)$, for all $p \geq 1$. In particular,

$$(6.3)\qquad < S^3(D_k(J))_d >_2= s(k,d).$$

Thus we have

$$(6.4)\qquad < S^3(D_k(J))_d >_{10}= s(k,d) j_5(\beta_5(1+\mu_5^{2k+1}\mathfrak{b}_k(J)_5)).$$

Here are some examples for $p = 5$. All of these examples are atypical except perhaps the last. In the first of these examples, we compare our result with previous calculations.

The covers of 0-surgery along the trefoil.

By (5.10) the eigenvalues of $Z_5(RT)$ are primitive 15th roots of unity. Thus $< S^3(RT))_d >_5$ is periodic with period fifteen. Actually the first fifteen values are $-A^4, A^3, 2\,A^2, A, -1, -2A^{-1}, A^3, -A^2, -2\,A, -1, A^{-1}, -2\,A^3, -A^2, A, 2$. Since the monodromy of the trefoil has order six, one might, at first, expect periodicity of order six. However by (5.4), $< S^3(RT) >_5= \kappa^{-3\sigma_7(RT)} < (S^3(RT))_7 >_5= -A^4$. This is consistent with our previous calculation of $< S^3(LT) >_5$. $S^3(RT)_6$ is the three torus. Thus the invariant of the three torus equipped with a p_1-structure α with $\sigma(\alpha)$ zero is $\kappa^{-3\sigma_6(RT)} < (S^3(RT))_6 >_5= \kappa^{24}(-2A^{-1}) = 2$. This calculation agrees with (3.9).

The covers of 0-surgery along the untwisted double of the figure eight knot.

$< S^3(K)_d >_5$ is given by the sum of the dth powers of the roots of $\Gamma_5(K)$ counted with multiplicity. It may also be easily computed recursively. This example is atypical in that we noticed something systematic. $\Gamma_5 D_0(F8) = x^2 - (A^2)x + (-3 + 2A - 2A^2 + 3A^3)$. Let $\lambda = 12(A+\bar{A}) - 8(A^2+\bar{A}^2) - 1$. λ is a positive real number under all complex embeddings of κ_5. In fact if A goes to $e^{\pm\frac{\pi i}{5}}$, then λ goes to approximately 13.4721. If A goes to $e^{\pm\frac{3\pi i}{5}}$, then λ goes to approximately 4.52786. Let κ be the positive square root of λ. Then the roots of $\Gamma_5\,(D_0(F8))$ are $\frac{1+\kappa i}{2}$, and $\frac{1-\kappa i}{2}$. So we have:

(6.5)

$$< (S^3(D_0(F8)))_d >_5= \left(\frac{A^{2d}}{2^d}\right)((1+\kappa i)^d + (1-\kappa i)^d) = \frac{A^{2d}}{2^{d-1}}\sum_{r=0}^{[d/2]}(-1)^r\binom{d}{2r}\lambda^r$$

The behavior of the argument (or phase) of $< (S^3(D_0(F8)))_d >_5 \bmod \pi$ as a function of d is quite simple in this case.

The covers of 0-surgery along the knot 8_1.

This example is more typical. The 3-twisted double of the unknot is 8_1. One may of course easily compute $< (S^3(8_1))_d >_5$ exactly by recursion. For example $< (S^3(8_1))_{17} >_5 = 188 + 152A + 136A^2$. There is not much pattern. However if we embedd k_5 in $\mathbb{C}$ by sending A to $e^{\frac{\pi i}{5}}$ then the eigenvalues of $Z_5(8_1)$ are $e_1 \approx 0.676766 - 1.2548i$, and $e_2 \approx .632251 + 0.303739i$. e_1 has norm greater than one and e_2 has norm less than one. Thus $< (S^3(8_1))_d >_5 = (e_1)^d + (e_2)^d$. So $< (S^3(8_1))_d >_{5_{|A=e^{\frac{\pi i}{5}}}} \approx (e_1)^d$, for d large.

§7 A connected sum formula

If c is a q-color, let $K(c)$ denote $S(K)$ with the image of a meridian colored c. Here and below the banding on a meridian is taken to be that given by another nearby meridian. As before we have a generator $\chi \in H^1(S(K))$. Let $Z_p(K,i) = Z_p((K,i),\chi)$. Define Γ_p similarly. Note $Z_p(K,0) = Z_p(K)$. Since V_p of a surface with a single odd colored banded point is zero,

Proposition (7.1). *If c is odd, $Z_p(K,c) = 0$*

Because V_p of a 2-sphere with a single colored banded point is zero, we have:

Proposition (7.2). *If $c \neq 0$, $Z_p(U,c) = 0$*

Let $U(i,j,k)$ denote zero surgery to the unknot with the images of three meridians colored i, j, and k, where these are are good q-colors. By [BHMV1,(4.4)], V_p of a 2-sphere with three points colored i, j, and k, is one dimensional if (i,j,k) is a small admissible triple, and is zero otherwise. Thus

Proposition (7.3). *Assume $p \geq 3$. $Z_p(U,i,j,k)$ is the identity on a free rank one k_p-module if (i,j,k) is a small admissible triple, and is zero otherwise.*

By (7.3), and the Colored Splitting Theorem [BHMV1,(1.14)], one has:

Theorem 7.4. *Assume $p \geq 3$.*

$$Z_p(K_1\#K_2) = \bigoplus_{i \text{ is a } q\text{-color}} Z_p(K_1,i) \otimes Z_p(K_2,i).$$

More generally:

$$Z_p(K_1\#K_2,i) = \bigoplus_{j,k \text{ such that } (i,j,k) \text{ is a small admissible triple}} Z_p(K_1,j) \otimes Z_p(K_2,k).$$

Proposition(7.5). *$Z_5(D_k(U),2)$ is periodic in k with period five. In particular,*

$$Z_5(D_{5n}(U),2) = [0]_\natural$$

$$Z_5(D_{5n+1}(U),2) = [1]_\natural$$

$$Z_5(D_{5n+2}(U),2) = [1 - A^3]_\natural$$

$$Z_5(D_{5n+3}(U),2) = [1 - A - A^3]_\natural$$

$$Z_5(D_{5n+4}(U),2) = [-A^2]_\natural$$

Proof. One may use the same basic procedure described in §5 to calculate the colored invariants of a knot. One only needs to add a straight colored line to the slab. V_5 of a torus with one framed point colored 2 is one dimensional [BHMV1,(4.14)]. A generator is pictured in Figure 7. To calculate $Z_5(D_k(U),2)$ one should attach the solid handlebody of Figure 7 to the bottom of the slab of Figure 3 with a vertical line colored 2 added so that the arcs labelled 2 match up. This new picture represents some multiple of the generator pictured in Figure 7. Just as in §5 we must multiply by κ^{-3} to correct for the p_1-structure. Recall $\omega = \eta(1+\delta z)$. But this diagram with the curve labelled ω deleted will represent zero since f_2 times $\supset\subset$ is zero in the Temperley Lieb algebra. Thus $Z_5(E)$ is multiplication by $\eta\delta\mu^{2k+1}$ times a_k, where a_k is the multiple of Figure 7 represented by our picture with curve label ω colored 1 and the $2k+1$ twists deleted.

One may calculate a_k doing a standard Kauffman bracket calculation of the link in the diagram colored 1, discarding any terms in an expansion where the segment labelled 2 is joined to a loop which is inessential (again since f_2 times $\supset\subset$ is zero.) One is left with a_k times the generator. One has $a_k = A^2 a_{k-1} + (1+A)\mu^{2-2k}$, and $a_0 = 0$. The result follows easily. □

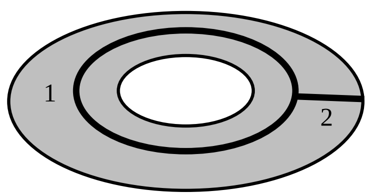

Figure 7

The above result may also be derived from (7.7),(7.9) and (7.10). It also follows from (8.6) below. The above proof is more elementary and helps us understand these invariants concretely. By (5.10),(7.4) and (7.5), we have for instance:

$$\mathcal{E}_5(RT\#LT) = \{1,1,1,A_3^2,\bar{A}_3^2\}$$
$$\mathcal{E}_5(F8\#F8) = \{1,1,1,A^2,\bar{A}^2\}$$
$$\mathcal{E}_5(RT\#RT) = \{A^4,\bar{A}^2,\bar{A}^2,A_3^2\bar{A}^2,\bar{A}_3^2\bar{A}^2\} \tag{7.6}$$

Before we made this calculation, we had found $\mathcal{E}_5(F8\#F8)$ using the method used in calculating $\Gamma_5(8_8)$.

Let $\mathcal{S}(c,p)$ be the set of good q-colors i such that (i,i,c) form a small admissible triple. The colored graphs in the solid torus pictured in Figure 7 with 1 replaced by i and 2 replaced by c as i ranges over $\mathcal{S}(c,p)$ forms a basis for V_p of a torus with a single point colored c, [BHMV1,(4.11)]. Let $n(c,p)$ be the cardinality of $\mathcal{S}(c,p)$. We form two $n(c,p)\times n(c,p)$ matrices $\mathsf{B}(J,k,c)$ and $\mathsf{L}(J,c)$ with rows and columns indexed by $\mathcal{S}(c,p)$. At this point, we begin to suppress the dependence on p again. $\mathsf{B}(J,k,c)_{i,j} = \beta\sum_{s=0}^{n-1} < e_s > \mu(s)^{2k+1} b(J,k,c)_{i,s,j}$ where $b(J,k,c)_{i,s,j}$ is

the bracket polynomial of the colored banded link in Figure 8a.

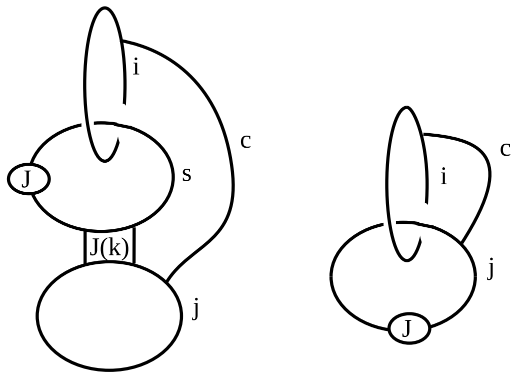

Figure 8a Figure 8b

Let $\mathsf{L}(J,c)_{i,j}$ be the bracket polynomial of the colored banded link in Figure 8b. Let $J(e)$ denote the knot J colored e.

Theorem (7.7). *If $p \geq 3$, and $< J_e >_p$ is nonzero for all $e \in \mathcal{S}(c,p)$, then $Z_p(D_k(J),c) = (\mathsf{B}(J,k,c)\mathsf{L}(J,c)^{-1})_\natural$.*

By the same arguments as used for (5.14), we have :

Corollary (7.8). *If $p \geq 3$, and $< J_e >_p$ is nonzero for all $e \in \mathcal{S}(c,p)$, then $Z_p(D_k(J),c)$ is periodic in k with period p.*

Using the tetrahedron coefficient $\left\langle \begin{matrix} A & B & E \\ D & C & F \end{matrix} \right\rangle$, in the notation of [MV2] , we have:

(7.9)

$$\mathsf{L}(U,c)_{i,j} = \sum_{r \ni (i,r,j) \text{ is a small admissible triple}} \frac{\delta(r;i,j)^2 < r >}{< r,i,j >} \left\langle \begin{matrix} c & j & j \\ r & i & i \end{matrix} \right\rangle .$$

(7.10)

If (c,s,s) is not a small admissible triple, $b(U,k,c)_{i,s,j} = 0$, otherwise $b(U,k,c)_{i,s,j} =$

$$\sum_{\substack{r,r' \ni (i,r,s), \text{ and } (j,r',s) \\ \text{are small admissible triples}}} \frac{\delta(r;i,s)^2 < r > \delta(r';j,s)^{2k} < r' >}{< r,i,s >< r',j,s >< c,s,s >} \left\langle \begin{matrix} c & i & i \\ r & s & s \end{matrix} \right\rangle \left\langle \begin{matrix} c & j & j \\ r' & s & s \end{matrix} \right\rangle .$$

§8 Quantum invariants of branched cyclic covers of knots

Let $s_d(K,c)_p$ denote the sum of the dth powers of the roots of $\Gamma_p(K,c)$. Note that in §6, we calculated $s_d(K,0)_p$ in several cases. The same methods, originally discussed in the remarks following (1.8), may be applied to calculate $s_d(K,c)_p$, from $\Gamma(K,c)$.

Proposition (8.1). *If $p \geq 3$ and c is even, $s_d(K,c)_p =< K(c)_d >_p$.*

Let K_d denote the branched cyclic cover of a knot K equipped with a p_1-structure α with $\sigma(\alpha) = 3\sigma_d(K)$. This is in the homotopy class of p_1-structures which extend across the branched cover of D^4 along a pushed in Seifert surface. Recall $< e_i >= (-1)^i \frac{A^{2i+2} - A^{-2i-2}}{A^2 - A^{-2}}$.

Theorem (8.2). *For $p \geq 3$,*

$$< K_d >_p = \begin{cases} \eta \sum_{i=0}^{(p-3)/2} < e_{2i} > s_d(K, 2i)_p & \text{if } p \text{ is odd} \\ \eta \sum_{i=0}^{[p/4]-1} < e_{2i} > s_d(K, 2i)_p & \text{if } p \text{ is even.} \end{cases}$$

Proof. Note that 0-framed surgery to $S^3(K)_d$ along the inverse image of the meridian is actually K_d. The trace of this surgery has zero signature, so the σ invariant of the p_1-structure is the same for K_d and $S^3(K)_d$.

Suppose we have a surgery description of K in S^3. Consider the resulting surgery description of $S^3(K)_d$. Let $\mathcal{D}_c$ be the result of replacing each surgery curve by ω with the given framing and then adding to the resulting picture the inverse image of the banded meridian colored c. Let $\mathcal{D}_\omega$ be the result of replacing each surgery curve by ω with the given framing and replacing the inverse image of the banded meridian by ω. We have that $< K(c)_d >_p = \eta < \mathcal{D}_c >$, and $< K_d >_p = \eta < \mathcal{D}_w >$. Thus by (8.1) if $p \geq 3$ and c is even, $< \mathcal{D}_c > = \eta^{-1} s_d(K, c)$. Here $< X >$ just the Kauffman bracket of the linear combination of framed links X, after letting $A = A_p$. Of course instead of replacing a curve by ω, we could replace it by any other combination of framed links in $S^1 \times B^2$ which represents the same element of $V_p(S^1 \times S^1)$.

If p is even, and c is odd, then by (3.1) $< K(c)_d >_p = 0$. If p is even, we are done since $\omega = \eta\Omega$, and $\Omega = \sum_{i=0}^{(p-4)/2} < e_i > e_i$. For p odd, we replace ω by $\omega' = \eta \sum_{i=0}^{(p-3)/2} < e_{2i} > e_{2i}$. One uses [BHMV2,6.3(iii)], to see that ω and ω' represent the same element in V_p of the boundary of a solid torus. $\square$

If K is a knot in S^3, then K_1 is S^3. Thus we have the following restriction on the colored invariants of K.

Corollary (8.3). *For $p \geq 3$,*

$$1 = \begin{cases} \sum_{i=0}^{(p-3)/2} < e_{2i} > \mathit{Trace}\ Z_p(K, 2i) & \text{if } p \text{ is odd} \\ \sum_{i=0}^{[p/4]-1} < e_{2i} > \mathit{Trace}\ Z_p(K, 2i) & \text{if } p \text{ is even.} \end{cases}$$

Since $\eta_3 = -1$, we have the following corollary which may also be derived from (5.3).

Corollary (8.4). $< K_d >_3 = -1$

Corollary (8.5). $< K_d >_6 = \eta_6 < S^3(K)_d >_6$, *and* $< K_d >_2 = \eta_2 < S^3(K)_d >_2$. *In particular,* $< (D_k(J))_d >_2 = \eta_2 s(d, k)$. *Thus* $\eta_{10}^{-1} < (D_k(J))_d >_{10} = s(d, k) j_5(\eta_5^{-1} < (D_k(J))_d >_5)$.

Proof. The first equation is just (8.2) for $p = 6$. Applying [BHMV1, (1.5)] to S^3 with $\sigma(\alpha) = 0$ we see $i_p(\eta_2) j_p(\eta_p) = \eta_{2p}$. The second equation follows from the first, $i_3(\eta_2) = -\eta_6$, and (8.4). The third equation follows from the second and (6.3). The last equation follows from the third and [BHMV1, (1.5)] again. $\square$

We may use (8.3) to obtain a generalization of (7.5). We note that when p is five, $< e_2 >^{-1} = -(A + \bar{A})$.

Corollary (8.6). $Z_5(D_k(J),2) = [(A+\bar{A})(\beta(1+\mu^{2k+1})\mathfrak{b}_k(J)-1)]_\natural$. *In particular, $Z_5(D_k(J),2)$ is periodic in k with period five.*

For instance $Z_5(D_0(F8),2) = [A^3 + A^4]_\natural$. In general, it is now an easy matter to calculate $< D_k(J)_d >_5$ recursively once we know $< J >$, and $[[J]]$. But these are just two values of the Kauffman polynomial, for which extensive tables exist. Moreover $< D_k(J)_d >_5$ is periodic in k with period five. We note that $D_2(U)$ is the stevedore's knot 6_1. We have for instance:

$$\eta^{-1} < RT_d >_5 = (A_3\bar{A})^d + (\bar{A}_3\bar{A})^d + (-1)^d(1 - A + A^4)A^{2d}$$
$$\eta^{-1} < F8_d >_5 = A^d + \bar{A}^d + (1 - A + A^4)$$
$$\eta^{-1} < (6_1)_d >_5 = 1 + \bar{A}^d + (1 - A + A^4)(1 - A^3)^d$$
$$\eta^{-1} < (8_1)_{17} >_5 = 1175 + 762A + 1123A^2$$
$$\eta^{-1} < (D_0(F8))_d >_5 = (1 - A + A^4)(A^3 + A^4)^d + \frac{A^{2d}}{2^{d-1}}\sum_{r=0}^{[d/2]}(-1)^r\binom{d}{2r}\lambda^r.$$

Here $\lambda = 12(A + \bar{A}) - 8(A^2 + \bar{A}^2) - 1$, as in (6.5). Note that $< F8_d >_5$ is periodic in d with period ten. Also $< RT_d >_5$ is periodic in d with period thirty. This second periodicity is generalized below. Let θ_p denote the invariant of oriented closed 3-manifolds defined in [BHMV2].

Theorem (8.7). *Suppose K is a fibered knot with a periodic monodromy of order s. Then $< K_d >_2$ is periodic in d with period s. If $p \geq 3$, then $< K_d >_p$, and $\theta_p(K_d)$ are periodic in d with period ps. If $r \geq 3$, $\tau_r(K_d)$ is periodic in r with period $2rs$. If $p = 2r$ with r odd and $r \geq 3$, then $< K_d >_p$ and $\theta_p(K_d)$ are periodic in d with period rs. If r is odd and $r \geq 3$, then $\tau_r(K_d)$ is periodic in r with period rs.*

Proof. The diffeomorphism type of $S^3(K)_d$ is periodic in d with period s. It follows that the first Betti number $b_1(< K_d >) = b_1(S^3(K)_d)$ is periodic with period s. By [DK,(5.2)], $\sigma_{s+k}(K) = \sigma_k(K) + \sigma_{s+1}(K)$. By Lemma (5.5), $\sigma_{s+1}(K) \equiv 0 \pmod 8$, so $\kappa^{9\sigma_{s+1}(K)} = u^{\frac{3}{2}\sigma_{s+1}(K)}$ is also a pth root of unity. Thus $\kappa^{3\sigma<K_d>}$ is periodic in d with period ps. In fact if p is even, $\kappa^{3\sigma<K_d>}$ has period $\frac{ps}{2}$. The first statement then follows from (8.5).

Now suppose $p \geq 3$. By (8.2), $< K_d >_p$, will be periodic in d with period ps if the roots $\Gamma(K,2i)_p$ are all psth-roots of unity. The underlying manifold of $K(2i)$ is the mapping torus of the closed off monodromy on the fiber capped off, which we denote Σ. Σ has been given the structure of a banded point colored $2i$, and $K(2i)$ has been given the extra structure of a meridian colored $2i$ with a certain banding. Let E be the associated fundamental domain of the infinite cyclic cover of $K(2i)$ and $E_s = \cup_{0\leq i \leq s-1} T^i E$ as in §1. $(Z_p(E))^s = Z_p(E_s)$ is then given by a scalar multiple of the identity. This is because E_s is diffeomorphic to $\Sigma \times [0,1]$, forgetting extra structure. Note that $K(2i)_s$ has an induced p_1-structure with σ equal to $3\sigma_d(K)$. Whereas if we gave $\Sigma \times [0,1]$ the product p_1-structure, then $Z_p(\Sigma \times [0,1])$ would be the identity, and the associated mapping torus has a p_1-structure with σ equal to zero. See the proof of (6.10). Also the banding on the links in E_s and $K(2i)_s$ differs by some number b of twists. Thus $(Z_p(E))^s$ is $\kappa^{9\sigma_s(K)}\mu(2i)^b$ times the identity. Note $\mu(2i)$ is a pth-root of unity. By lemma (5.5), $\sigma_s(K) \equiv 0 \pmod 8$.

So $\kappa^{9\sigma_s(K)} = (u^2)^{3\sigma_s(K)/4}$ is also a pth root of unity. It follows that $(Z_p(E))^{ps}$ is the identity and all the roots of $\Gamma(K,2i)_p$ are psth-roots of unity.

By [BHMV1,§2], we have that $\theta_p(K_d)$ are periodic in d with period ps. By [BHMV3,(2.2)], $\tau_r(K_d)$ is periodic in r with period $2rs$.

If $p = 2r$ where r is odd use [BHMV1,(1.5)] to express $< K_d >_p$ in terms of $< K_d >_r$ and $< K_d >_2$. This gives the above periodicity for $< K_d >_p$. The periodicity of $b_1(< K_d >_p)$, $\kappa^3\sigma < K_d >$, and [BHMV3,(2.2)] then yield the above periodicity of $\theta_p(K_d)$ and $\tau_r(K_d)$. □

Using Goldsmith's construction [Go] of the fibration for $T(a,b)$, the (a,b) torus knot, it is easy to see that the monodromy is periodic with period ab. $K(a,b)_c$ is the the Brieskorn manifold $\Sigma(a,b,c)$ with a p_1-structure α such that $\sigma(\alpha) = 3\sigma_c(T(a,b)$. $\sigma_c(T(a,b))$ is equal to the signature of the variety $z_0^a + z_0^b + z_0^c = 1$ in $\mathbb{C}^3$. There is a well known formula due to Brieskorn for this signature. Our convention is that $\Sigma(a,b,c)$ is oriented as the boundary of the variety $z_0^a + z_0^b + z_0^c = 1$ intersected with the 6-ball. We have that

$$< \Sigma(a,b,c) >_p = < \Sigma(a,b,c+pab) >_p$$

$$\tau_r(\Sigma(a,b,c)) = \tau_r(\Sigma(a,b,c+2rab)) \text{ if } r \text{ is even} \tag{8.8}$$

$$< \Sigma(a,b,c) >_{2r} = < \Sigma(a,b,c+rab) >_{2r} \text{ if } r \text{ is odd}$$

$$\tau_r(\Sigma(a,b,c)) = \tau_r(\Sigma(a,b,c+rab)) \text{ if } r \text{ is odd} \tag{8.9}$$

Making use of the fact $\Sigma(a,b,-c) = -\Sigma(a,b,c)$, one sees that these equations hold for all a,b,c positive or negative. In this way one obtains four further relations, for example:

$$< \Sigma(a,b,c) >_p = \overline{< \Sigma(a,b,pab-c) >_p}.$$

The fact that the diffeomorphism type of $\Sigma(a,b,c)$ is invariant under permutations of (a,b,c) leads to further relations among $< \Sigma(a,b,c) >_p$.

Freed and Gompf showed, for c of the form $6k\pm1$, that $\tau_r(\Sigma(2,3,c))$ was periodic in c with period $6r$ in the case r odd and with period $3r$ in the case r even. They used the fact that $\Sigma(2,3,6k\pm1)$ is $(\mp1/k)$-surgery on $\pm T(2,3)$ and the periodicity of τ_r for $(1/n)$-surgery on a knot due to Kirby and Melvin. In (8.9) above, we generalize this periodicity for r odd. In (8.8), we get periodicity with four times this period. However we have no restriction on c in either case.

Let $v_p(g,c)$ be the dimension of V_p of a surface of genus g with a single point colored c. We may use (3.8),(8.2), and the triangle inequality to obtain:

Theorem(8.10). *If K is a fibered knot with genus g, then for all d*

$$|\mathfrak{i}\left(\eta^{-1}(< K_d >_p)\right)| \le \begin{cases} \sum_{i=0}^{(p-3)/2} \mathfrak{i} < e_{2i} > v_p(g,2i) & \text{if } p \text{ is odd} \\ \sum_{i=0}^{[p/4]-1} \mathfrak{i} < e_{2i} > v_p(g,2i) & \text{if } p \text{ is even.} \end{cases}$$

§9 Partial Invariance under skein equivalence

The invariants which we discuss are almost invariant under skein equivalence. To see this, we first define some cobordism categories with more morphisms. We define $\check{C}_2^{p_1}$ to be the cobordism category with objects those of $C_2^{p_1}$ but whose morphisms are k_p-linear combinations of morphisms of $C_2^{p_1}$ between the same objects.

Composition is defined as follows. Suppose $\{C_i\}$ is a finite set of morphisms from Σ_1 to Σ_2 and $\{C'_j\}$ is a finite set of morphisms from Σ_2 to Σ_3, then $\{C_i \cup_{\Sigma_2} C'_j\}$ is a finite set of morphisms from Σ_1 to Σ_3. Define

$$\sum_i a_i C_i \circ \sum_j a_j C'_j = \sum_{i,j} a_i b_j \ (C_i \cup_{\Sigma_2} C'_j).$$

We define Z_p on $\check{C}_2^{p_1}$ by extending linearly. One may define a expansion functor from $C_{2,q}^{p_1,c}$ to $\check{C}_2^{p_1}$, following the recipe in [BHMV1]. Similarly we define $\check{C}_{2,q}^{p_1,c}$ to be the cobordism category with objects those of $C_{2,q}^{p_1,c}$ but whose morphisms are k_p-linear combinations of morphism of $C_{2,q}^{p_1,c}$ between the same objects. We may define $Z_p(M,\chi)$ if M is a morphism in these new categories from the $\emptyset$ to $\emptyset$ just as in §3. Let $\mathcal{B}$ be a basis for $V(\Sigma)$, let $\mathbb{M}(M,\Sigma,\mathcal{B})$ denote the matrix for $\kappa^{-\sigma(\alpha(M))} Z_p(E(\Sigma)$ with respect to $\mathcal{B}$. Thus $Z_p(M,\chi) = (\mathbb{M}(M,\Sigma,\mathcal{B}))_\natural$.

Given a colored graph G in M transverse to a choice of Σ dual to χ, we may define the skein equivalence class of G modulo Σ to be the equivalence relation generated by ambient isotopies of G, which are the identity in a neighborhood of Σ and the local moves described in [MV2,§2] which takes place in the complement of this neighborhood. If G is really a link L (colored 1), one takes the usual Kauffman skein relation. We clearly have:

Proposition (9.1). *Let Σ be a fixed surface dual to χ, and $\mathcal{B}$ is a basis for $V_p(\Sigma)$, $\mathbb{M}(M,\Sigma,\mathcal{B})$ is an invariant of the skein equivalence class of G modulo Σ. Thus, if Σ is some fixed surface dual to χ, $Z_p(M,\chi)$ is well defined on the skein class of G modulo Σ.*

In a similar way, we have in the notation of §4:

Proposition (9.2). *$\mathcal{Q}(\mathcal{T})$ is an invariant of the skein equivalence class of $\mathcal{T}$.*

§10 $V_p((S^2,m)\coprod(S^2,1))$ and odd Links in $S^1\times S^2$

Here we discuss another approach to odd links. We let (S^2,m) demote a 2-sphere with m banded points (colored one). $K(S^2,m)$ is zero for m odd, so $V(S^2,m)$ vanishes as well. Whenever $V_p((S^2,m_1)\coprod(S^2,m_2))$ is nonzero, one may consider $z_p(L,m_2) = Z_p\left((S^1\times S^2,L)\coprod(S^1\times S^2,L_{m_2})\right)$, where L_m denotes m S^1- factors. As a first case, we consider $z_p(L,1)$. It turns out that the few invariants we get in this case are very trivial. However our calculation of $V_p((S^2,m_1)\coprod(S^2,1))$ may be of interest.

Lemma (10.1). *For $p\neq 3$ or 1, $V_p((S^2,m)\coprod(S^2,1)) = 0$. $V_3((S^2,m)\coprod(S^2,1)) = k_p$ if m is odd and is zero if m is even. $\dim(V_1((S^2,m)\coprod(S^2,1))) = c(\frac{m+1}{2})$ if m is odd and is zero if m is even.*

Proof. For p even and p greater than two, this follows since $V_p((S^2,1))=0$ and the tensor product axiom (M) holds [BHMV1,(1.10)]. According to [BHMV1,(1.9)], there is an epimorphism $\epsilon_{(I\times S^2,m,1)} : K(I\times S^2,m,1) \to V_p((S^2,m))\coprod(S^2,1)))$. Here $(I\times S^2,m,1)$ denotes $I\times S^2$ with m framed points on $\{0\}\times S^2$ and 1 framed points on $\{1\}\times S^2$. $K(I\times S^2,m,1)$ is trivial if m is even, this proves the result if m is even. From now on we assume that m is odd and p is either odd or equal to two.

Now $K(I \times S^2, m, 1)$ also has a basis consisting of the $c(m+1)$ diagrams $\mathcal{E}_i$ in $I \times B^1$ with no crossing. We define a bijection f from this basis to the corresponding basis for $K(D^3, m+1)$ by wrapping the segment that meets the isolated endpoint back around "to the left" so that all endpoints are on the same side as in Figure 9.

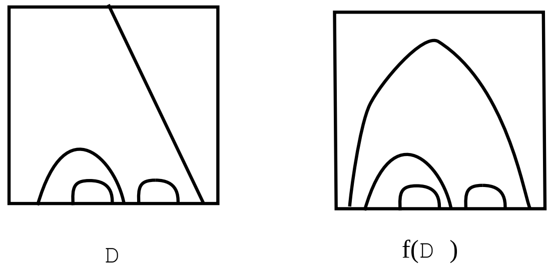

Figure 9

As before we define a matrix $D_p(m,1)$ with entries $< \mathcal{E}_i, \mathcal{E}_j >_p$. This is calculated from the $<, >_p$ -invariant of the pair $(S^1 \times S^2, \mathcal{E}_i \cup -\mathcal{E}_j)$, where $\mathcal{E}_i \cup -\mathcal{E}_j$ is given by $\mathcal{E}_i$ glued along m framed points to the mirror image of $\mathcal{E}_j$ with the other points joined up in a straight fashion but traveling around the S^1 factor in $S^1 \times S^2$. See Figure 10 where we have drawn the link in $S^1 \times S^2$ that we obtain when we pair the diagram of Figure 9 with itself.

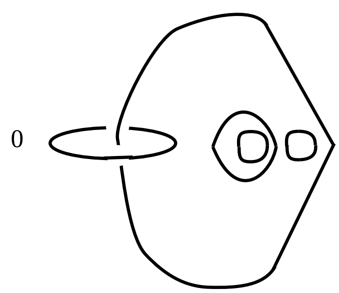

Figure 10

We may evaluate $< \ >_p$ of the pair by taking the bracket polynomial of a linear combination of banded links evaluated at $A = A_p$. The specific linear combination is obtained by replacing the component labelled zero by the linear combination of banded links given by ω as described in [BHMV1,5.8,§2]. Let H be the bracket polynomial of a Hopf link with one component replace by ω. Note in our pictures the framing of a link is the "blackboard framing". It is clear that $< \mathcal{E}_i, \mathcal{E}_j >_p$ is $\frac{H}{\delta}$ times $f(\mathcal{E}_i)$ paired with $f(\mathcal{E}_j)$ evaluated at $A = A_p$. Thus $D_p(m,1) = (\frac{H}{\delta} D(m+1))_p$. Here and above, the bracket is normalized by saying that the bracket of the empty link is one.

We now observe that H_p is zero for p not equal to one, or three. It will follow that $V_p((S^2, m)) - (S^2, 1))) = 0$ for $p = 2$ and for p odd and greater than three. This must be true by the "Dirac string trick". See the description of this trick [K4,(p.9)], which seems to be a precursor to the "light bulb trick" [R,(p.257)]. If H_p were nonzero, the bracket polynomials of the unknot with writhe zero and with writhe two would be the same when evaluated at $A = A_p$. This can only happen if $A_p^6 = 1$. Thus $p = 1$ or $p = 3$.

In fact, it is easy to show using the axioms that for any link diagram the bracket evaluated at A_3 is one. Direct calculation shows $H_3 = 2$. Note that $\delta_3 = 1$, so

$D(n)_3$ is a matrix with one in every entry . It follows $\epsilon_{(I\times S^2,m,1)}(\mathcal{E}_i)$ for each i is the same nonzero element of $V_3((S^2,m))\coprod(S^2,1)))$. Direct calculation shows $H_1 = -2\kappa^3$, depending on our choice of $(\kappa^3)^2 = 1$. Note that $\delta_1 = -2$, so $D_1(m,1) = \kappa_1^3 D(m+1)_1$ which is nonsingular by [KS]. □

Corollary (of the proof) (10.2). *If m is odd, $\{\mathcal{E}_i\}$ is a basis for $V_1((S^2,m)\coprod(S^2,1))$.*

Remark According to [BHMV1,(3.9)] If p is odd and $p\geq 3$, then $V_p((S^2,p-2)\coprod(S^2,p-2)) = k_p$. For p=3, this agrees with the above.

Given a tangle diagram $\mathcal{T}$ with an odd number of stands, let $\mathcal{T}^+$ denote $\mathcal{T}$ with one extra vertical straight strand on the left. It is not hard to prove (10.3) below. One makes use of the fact the bracket polynomial evaluated at A_1 is just $\delta_1^{\#(D)}$, where $\#(D)$ denotes the number of components of D.

Theorem (10.3). *Suppose L is an odd link in $S^1\times S^2$. Then $z_3(L,1)$ is trivial. If $\mathcal{T}$ is a tangle diagram for L, then $z_1(L,1) = z_1(L(T^+))$. It follows that $z_1(L,1)$ only depends on the absolute values of the degrees of the individual components of L.*

Remark Subsequent to an earlier version of this paper, Basinyi Chimitza [C] has made some further calculations for p odd and $p\geq 3$. He has shown that Z_p on $C_{2,q}^{p_1,c}$ satisfies a generalized tensor product axiom in the sense of Blanchet and Masbaum [BM,Ma], with $\hat\Sigma$ given by a 2-sphere with a single banded point colored $p-2$. Also he has shown that for Σ connected $V_p((\Sigma,c)\coprod\hat\Sigma)$ is isomorphic to $V_p((\Sigma,c)\#\hat\Sigma)$. Moreover $V_p(\Sigma\coprod(S^2,m)) = 0$ if the sum of the colors of the points of Σ is odd and m is odd and $m<p-2$. Also $\dim(V_p((S^2,p-2)\coprod(S^2,p))) = p-2$, and $\dim(V_p((S^2,p)\coprod(S^2,p))) = (p-2)^2$.

§11 Comparison with other calculations

Using our exact calculations based on (5.13), we have computed the invariant $<\,>_p$ for zero surgery to S^3 along RT, F8, 5_2, 6_1, 7_2, for $p\leq 18$. For RT, F8, we also have computed $p=19$, and $p=20$. Numerical approximations of these calculations for $p=2r$ agree with the values given in [KL,Tables:4,15,43,47,62] for $r=p/2$. Using (5.2), we have used the calculations for p odd to approximate $<\,>_{2p}$. Again the values agree with the values given in [KL] for $r=p$. According to [KL], their values agree with those of Freed and Gompf [FG] , and those of Neil [N] wherever they overlap.

RT and $F8$ are genus one fibered knots. So one can calculate $\hat Z_p(RT)$ and $\hat Z_p(F8)$ using the representation of $SL(2,\mathbb{Z})$ specified by Witten. However this relies on knowing the equivalence of Witten's theory and the theory of [RT,KM,L3,BHMV1]. I am not sure that this has been completely established. Also we remark that for non-fibered knots our method may be the only systematic way to make these computations. We have attempted to calculate $\hat Z_p(RT)$ and $\hat Z_p(F8)$ using [FG] and Jeffrey [Je]. In order to get the results to agree with our earlier calculations, we needed to take an ordered basis for the homology of an oriented fiber so that the intersection pairing of the first basis element with the second was minus one. This is done for the left handed trefoil in [FG] without comment. Also we noted that the left hand side of equation [Je,(2.22)] should be multiplied by three so that $\delta(M,\pi)$ is integral. In view of [A3] ,[KM2] and the final comment in [Sc], we modified equation [Je,(4.4)] to read $\Psi(U) = -\Phi(U)+3\nu$. Here ν is in the notation of [KM2].

We also assumed that in the translation we should replace $e^{2\pi i/4r}$ by $-A_{2r}$ as in [L3,Prop8]. If K is either RT or $F8$, let $w_r(K)$ denote $e^{-2\pi i\Psi(U)}\mathcal{R}(U)$, after we have replaced $e^{2\pi i/4r}$by $-A_{2r}$, where U is a monodromy matrix for the fibering. By applying the formula [Je(2.7b)] for $\mathcal{R}$, and making use of the Gauss sum $g(p,1)$ of [BHMV1,§2], we obtain

(11.1)
$$w_r(RT) = \left(\frac{(-1)^{r+1}A^{4-r^2}}{4r} \sum_{m=1}^{4r} A^{-m^2} \right) [(-A)^{-l^2}(A^{2lj} - A^{-2lj})]_{j,l}$$

(11.2)
$$w_r(F8) = \left(\frac{(-1)^{r+1}A^{-r^2}}{4r} \sum_{m=1}^{4r} A^{-m^2} \right) [(-A)^{j^2+2l^2}(A^{2lj} - A^{-2lj})]_{j,l}$$

Here $1 \leq j \leq r-1$ and $1 \leq l \leq r-1$. The characteristic polynomials of $w_r(RT)$ and $w_r(F8)$ agree with our own calculations of $\Gamma_{2r}(RT)$ and $\Gamma_{2r}(F8)$ for $3 \leq r \leq 10$. We have also checked that $w_r(RT)$ and $w_r(F8)$ are periodic in this range. Thus $w_r(RT)$ is similar to $\hat{Z}_{2r}(RT)$, and $w_r(F8)$ is similar to $\hat{Z}_{2r}(F8)$ in this range. Of course, this is expected for all r. We note that $w_r(F8)$ is similar to $A^{-4}w_r(RT)[(-A)^{4l^2}\delta^l_j]_{j,l}$.

Using [BHMV3,(2.2)] and [BHMV1, (1.5)], one has for any closed 3-manifold M with p_1-structure α that

(11.3)
$$\tau_5(M) = \beta_{10}^{-1} v^{-9-3\sigma(\alpha)} < M >_{10} \ {}_{|A=-v^2 \text{and } \kappa = v^3}.$$

where $\beta_{10}^{-1} = \eta_{10}^{-1}\kappa^{-3} = -1 - A + A^2 - A^3 - A^4 + 2A^6$. Of course the left hand side does not depend on α.

We can compare our calculation of $(\eta_5)^{-1} < RT_d >_5$ with Freed and Gompf's calculation of Witten's invariant of level 3 for $\Sigma(2,3,c)$ and Neil's calculation of τ_5 for $\Sigma(2,3,c)$. Freed and Gompf calculate these for some $c = 6k+1$. However they observe that $\Sigma(2,3,6k-1) = \overline{\Sigma(2,3,1-6k)}$. This together with their periodicity and the values they calculate is enough to determine $\tau_5(\Sigma(2,3,6k \pm 1))$ for all k. Neil calculated $\tau_5(\Sigma(2,3,6k \pm 1))$ for $1 \leq k \leq 5$. In fact using (11.3), (8.5), and (5.4), we have:

$$\tau_5((D_k(J))_d) = A^{\frac{1}{2}\sigma_d(D_k(U))} s(k,d) j_5(\eta_5^{-1} < (D_k(J))_d >_5) \ {}_{|A=-e^{2\pi i/20}}$$

It turns out our results always agree with Freed and Gompf's. Our results agree with Neil's for $c = 6k-1$ and are the conjugates of Neil's when $c = 6k+1$. Presumably, this is due to a different choice of orientation conventions. Our orientation convention is the same as Freed and Gompf's.

§12 Afterword

Here are a few reasonable conjectures:

Conjecture 1 $\Gamma_p(K)$ has coefficients which are integral polynomials in A_p.

Conjecture 2 $Z_p(F8)$ is a periodic map.

Conjecture 3 $Z_p(RT)$ is a periodic map with period $3p$, for all p.

Conjecture 4 Γ_p(stevedore's knot) has one as a root.

Conjecture 5 The degree of Γ_{2r}(tweenie knot) is less than $r-1$.

Perhaps we can tackle Conjectures 2 and 3, using (11.1) and (11.2).

Of course there are many ways that we may begin with a familiar situation in link theory, and obtain a colored graph in a closed 3-manifold with a one dimensional cohomology class. For instance given a link in a homology sphere one may take a sublink and obtain a manifold M by performing surgery to each component of the sublink with framing minus the sum of the linking numbers with the other components of the sublink. Then $H^1(M)$ is free Abelian on the components of the sublink. The rest of the link may then be colored. In this way we obtain many invariants. As an example: perhaps we are interested in the symmetries of a 2-component link (K_1, K_2). One may do 0-framed surgery on K_1 and color K_2, and compare this with 0-framed surgery on K_2 with K_1 colored. Alternatively one could form M by performing surgery along both components and then determine if $Z_p(M,\chi) = Z_p(M, s\chi)$ where s is an involution on $H^1(M)$ switching duals to the meridians.

Appendix

By a list we mean an unordered finite collection with repetitions allowed. There is a bijection from the set of lists of elements of k to the set of polynomials in $f[x]$ given by $\mathfrak{P}(\{\lambda_1, \lambda_2, \dots \lambda_r\}) = \prod_i x - \lambda_i$. Given two lists we may take the list of products of pairs of entries from each list. Corresponding to this product on the set of lists there is a corresponding product, which we will also denote by $\otimes$. The following useful formulae and their generalizations for polynomials of higher degree are easily worked out:

$$(x + a_0) \otimes (x + b_0) = x - a_0 b_0 \tag{A.1}$$

$$(x + a_0) \otimes (x^2 + b_1 x + b_0) = x^2 - a_0 b_1 x + a_0^2 b_0 \tag{A.2}$$

$$\begin{aligned}(x^2 + a_1 x + a_0) \otimes (x^2 + b_1 x + b_0) = \\ x^4 - (a_1 b_1)x^3 + (a_0 b_1^2 + a_1^2 b_0 - 2a_0 b_0)x^2 - (a_0 a_1 b_0 b_1)x + a_0^2 b_0^2.\end{aligned} \tag{A.3}$$

The product $\otimes$ is closely related to the product $\boxtimes$ defined in [HZ,(p.34)].

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